Extreme Value Analysis

Systems Thinking,
Spring 2011
Boulder, Colorado

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Photo by Everett Nychka

Motivation

Colorado Lottery

POWER PLAY PAYOUT TABLE

MATCH	PRIZE	X2	X3	X4	X5
○○○○○●	Jackpot	POWER PLAY does not apply.			
○○○○○	$200,000	$1,000,000*			
○○○○●	$10,000	$20,000	$30,000	$40,000	$50,000
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$$\Pr\{\text{Winning } \geq \$10,000 \text{ in one drawing}\} \approx 0.000001306024$$

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In ten years, playing one ticket everyday, $\Pr\{\text{Winning } \geq \$10,000\} \approx 0.004793062$

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In 100 years ≈ 0.05003321

In 1000 years ≈ 0.7686185

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In 1000 years ≈ 0.7686185

Law of small numbers: events with small probably rarely happen, but have many opportunities to happen. These follow a ***Poisson distribution***.

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Can also talk about waiting time probability. The ***exponential distribution*** models this. For example, the probability that it will take longer than a year to win the lottery (at one ticket per day) is ≈ 0.999523, longer than ten years ≈ 0.9952411, longer than 500 years ≈ 0.7877987, and so on (decays exponentially, but with a very slow rate).

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Another way to put it is that the expected number of years that it will take to win more than $10,000 in the lottery (buying one ticket per day) is about 2,096 years. If a ticket costs $1, then we can expect to spend $765,682.70 before winning at least $10,000.

Motivation

Taleb, N.N. 2010: *The Black Swan: The impact of the highly improbable*, Random House, New York, NY, 444 pp.

Outline

- Further motivation for why *extremes* are of interest, and why they require careful attention to analyze them.
- Introduce the basics of statistical Extreme Value Analysis (EVA).
- Discuss some limitations for practical applications (climate heavy).
- Introduce the idea of correlation, and why this topic has caused a lot of controversy regarding the current economic crisis.

Motivation

On the eve of the events in 1914 leading to WWI, would you have guessed what would happen next?

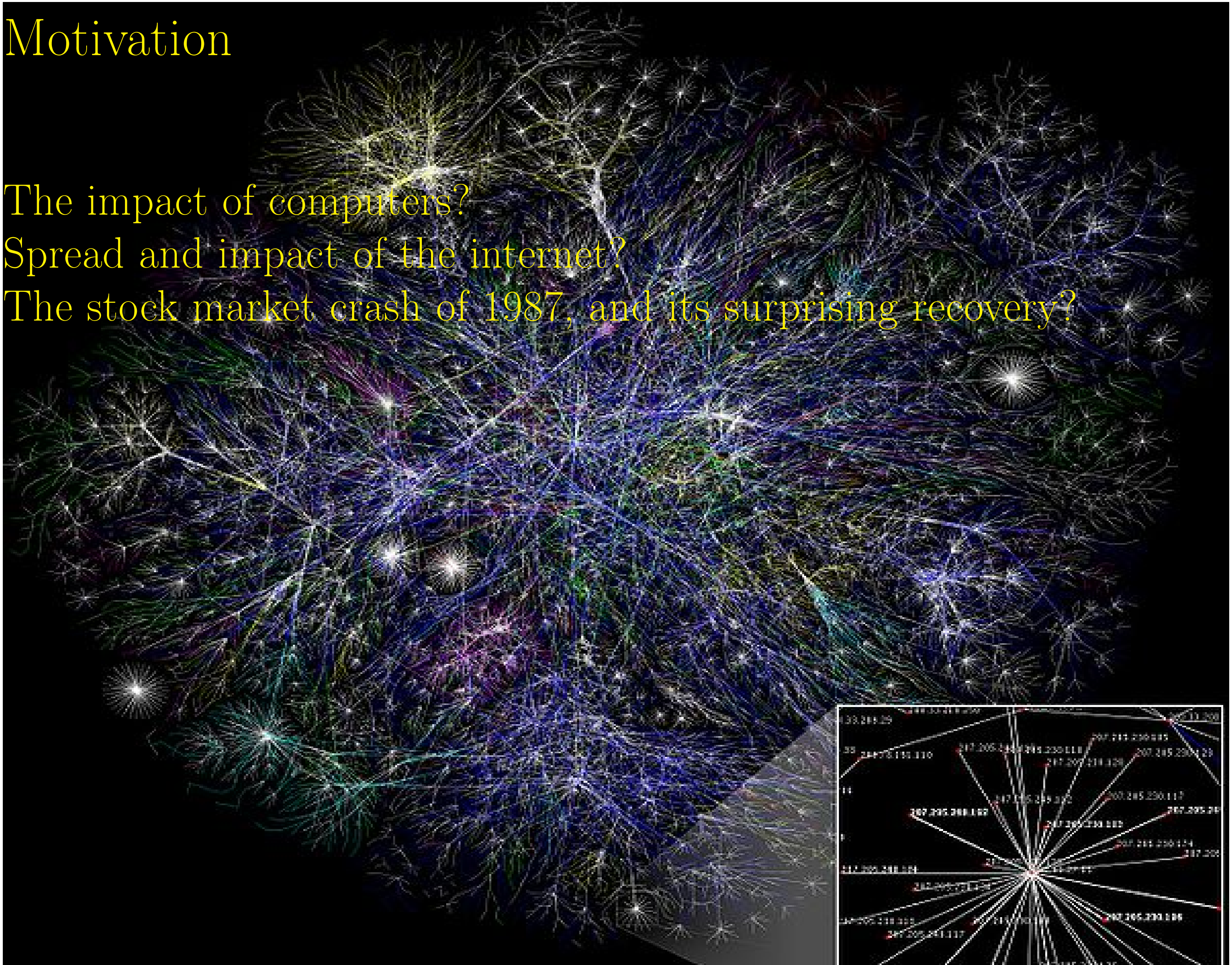
Archduke Franz Ferdinand of Austria

How about the rise of Hitler and WWII?

Adolf Hitler

Motivation

The impact of computers?
Spread and impact of the internet?
The stock market crash of 1987, and its surprising recovery?



Motivation

Retrospective Predictability

Different from Prospective Predictability. Once something has happened, it is easier to trace the steps to find the cause and effect.

Perspective

Insider Trading can lead to an extreme event that is well prospectively predicted by those on the inside, but if done right, is a surprise to everyone else (*ethics*).

Risk

Have you considered *extreme events* in your risk analysis for your financial portfolio?

Motivation

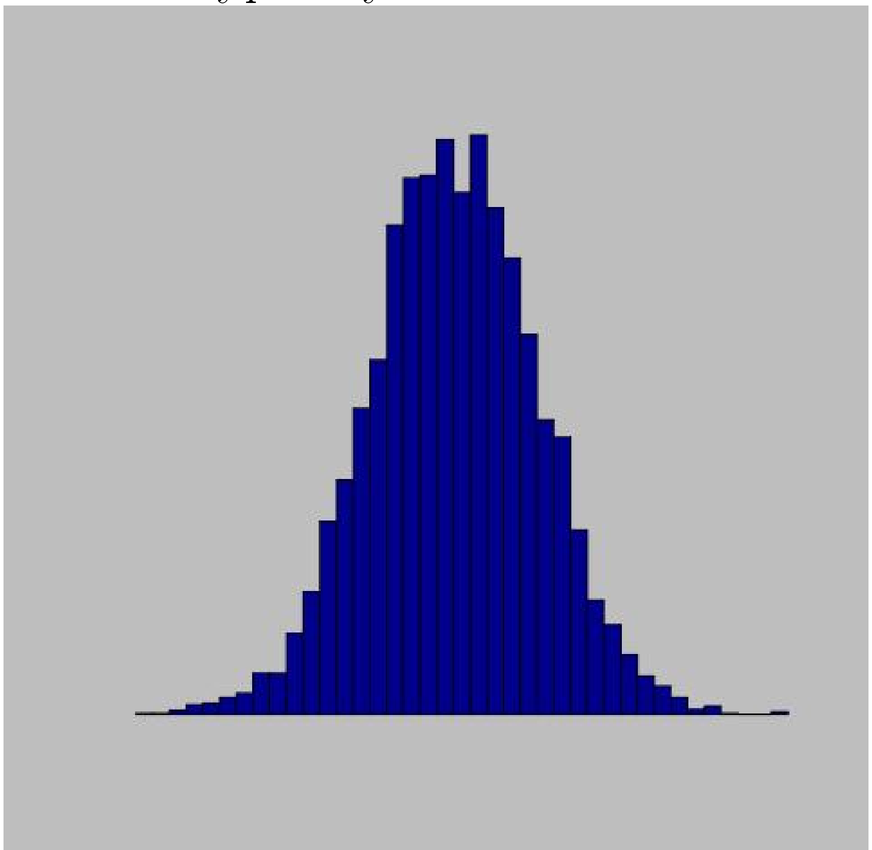
Taleb defines a Black Swan event as

- being rare
- having an extreme *impact*
- being predictable *retrospectively*, not prospectively.

Motivation

Randomness and Large Deviations

Focus is typically on *central tendencies*,



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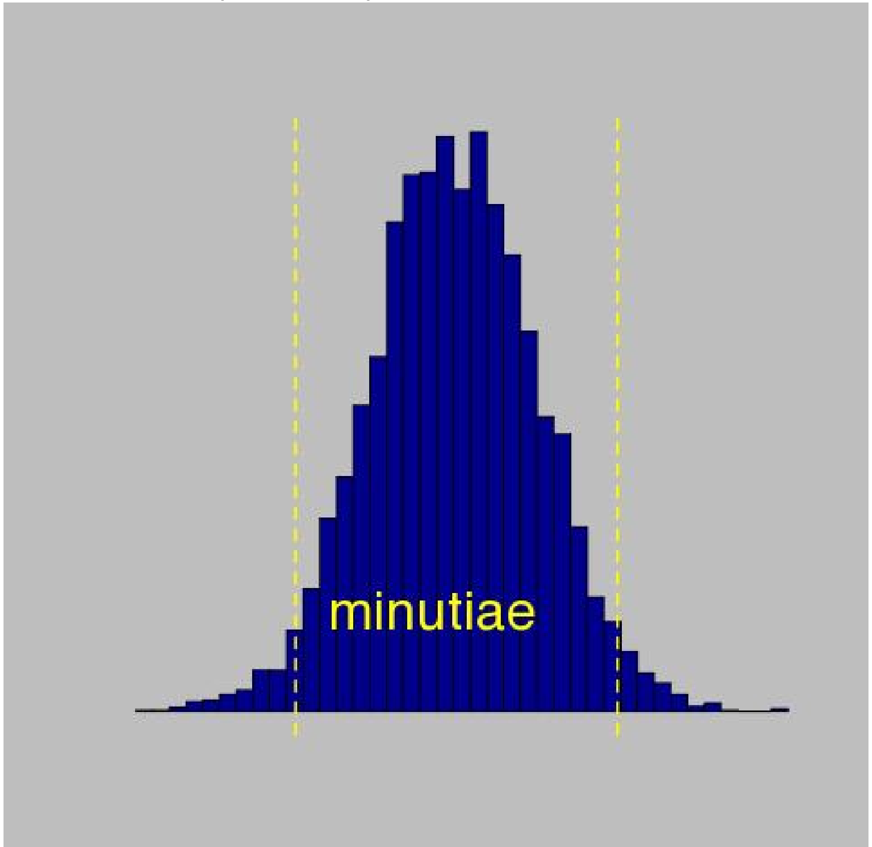

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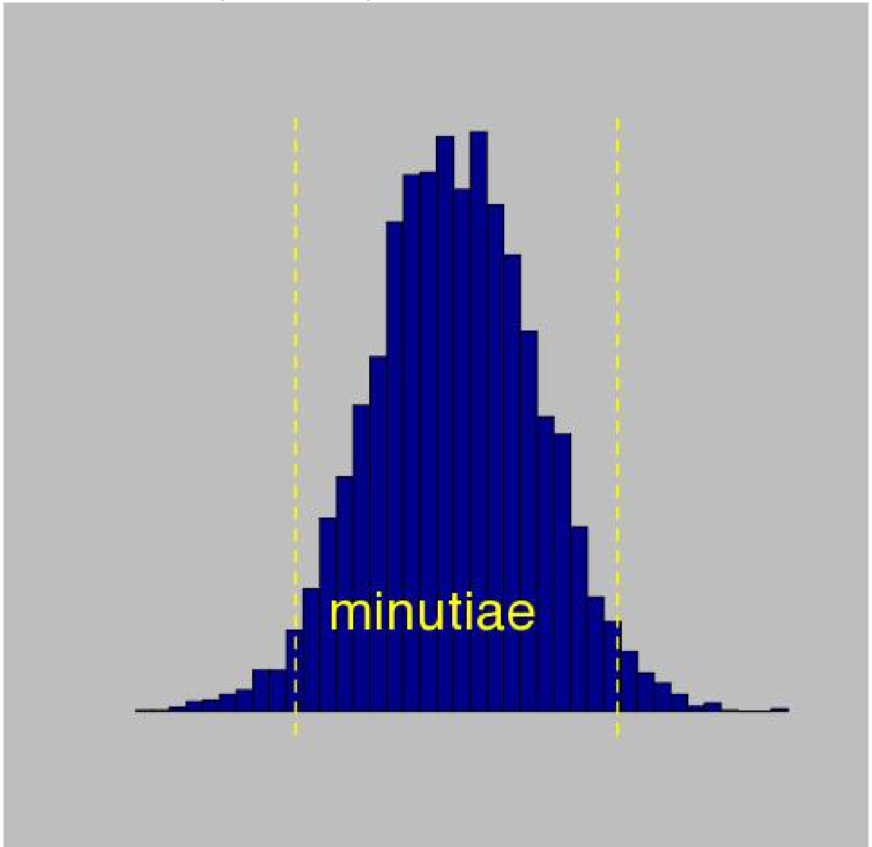
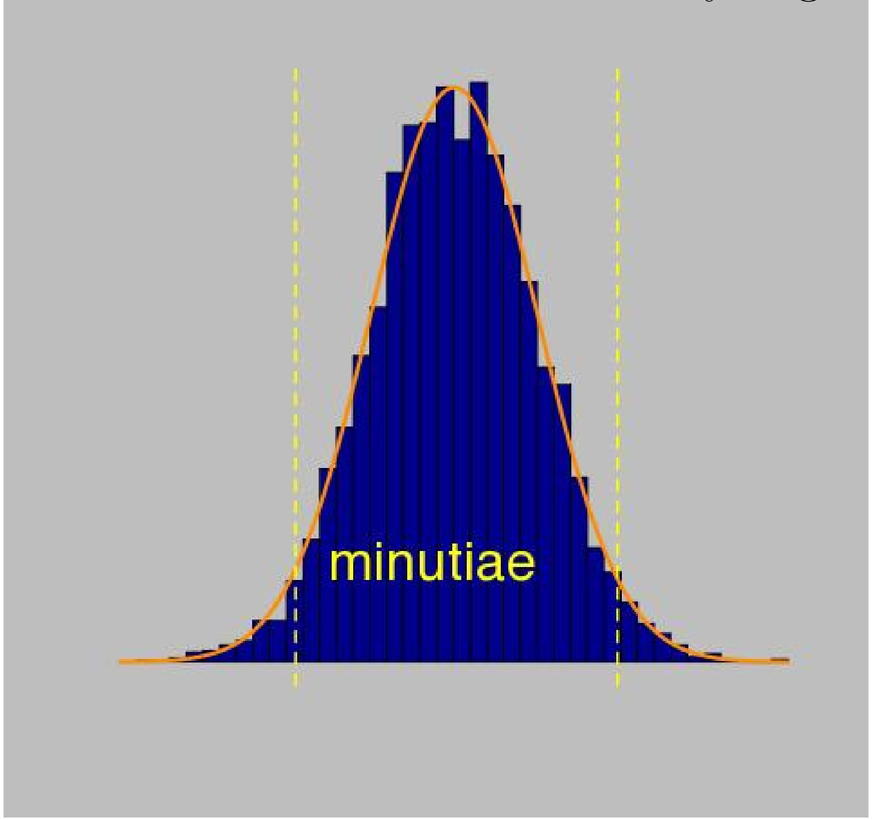


Motivation

Law of Large Numbers, Sum Stability, Central Limit Theorem

And other results give theoretical support for use of the

Normal distribution for analyzing most data.



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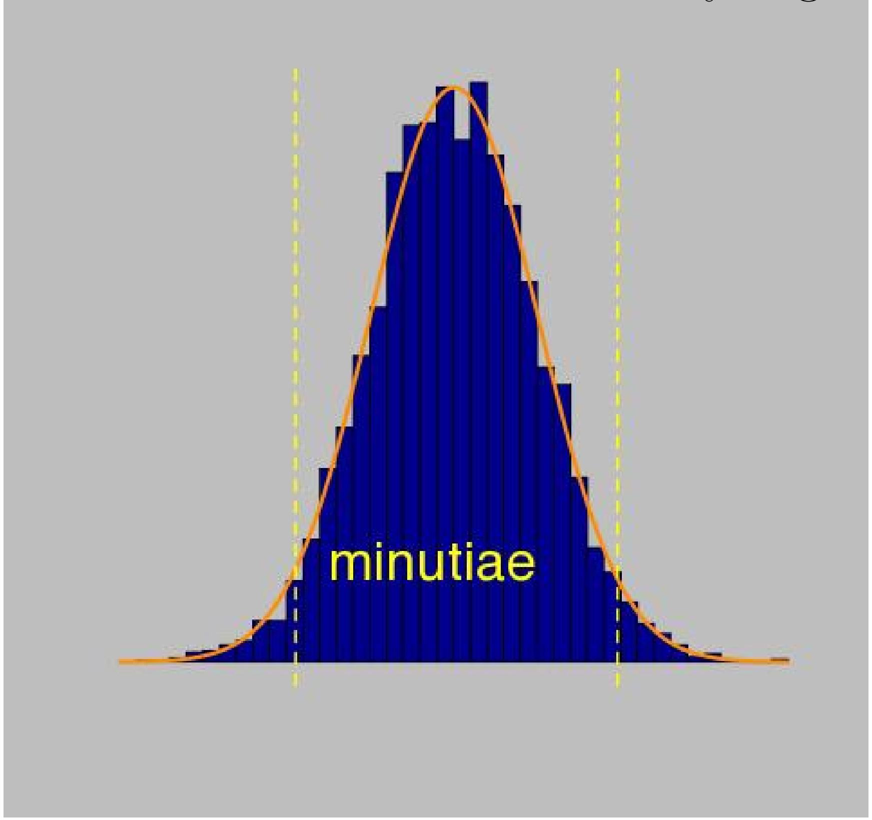

But, it is the ***possible*** extreme (or rare) events that are the most ***influencial***.

Extremal Types Theorem

Theoretical support for using the **Extreme Value Distributions (EVD's)** for ***extrema***.

- Valid for maxima over very *large* blocks, or
- Excesses over a very *high* threshold.

It is possible that there is no appropriate distribution for extremes, but if there is one, it must be from the Generalized Extreme Value (GEV) family (block maxima) or the Generalized Pareto (GP) family (excesses over a high threshold). The two families are related.

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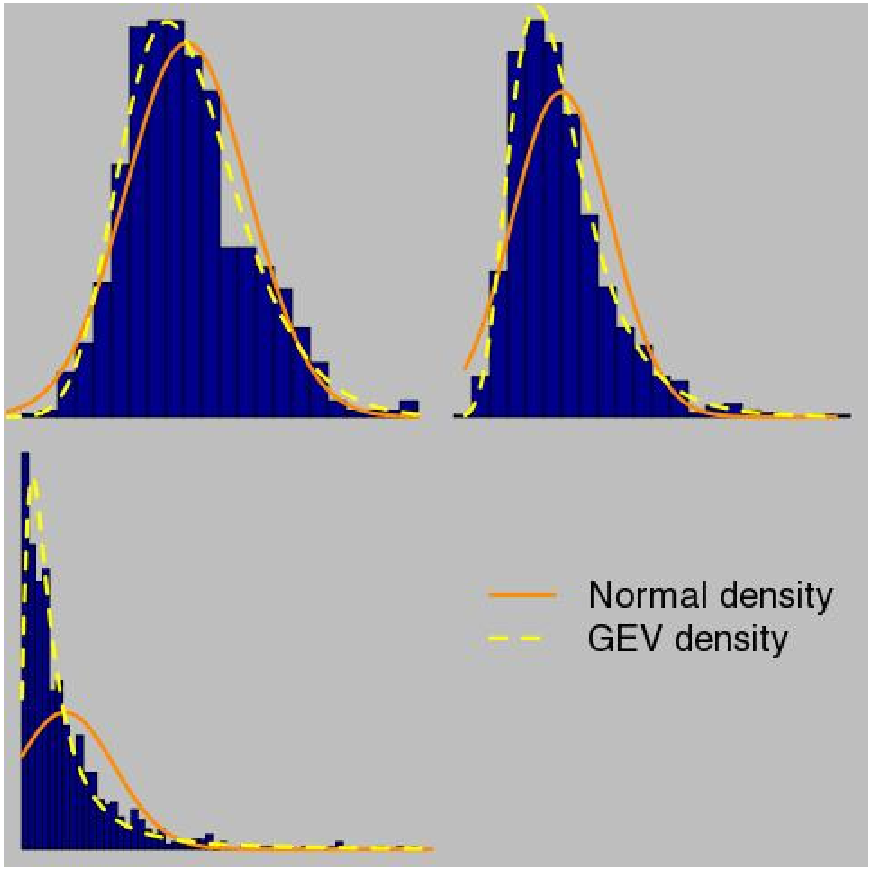
Poisson process allows for a nice characterization of the threshold excess model that neatly ties it back to the GEV distribution.

Background

Simulated Maxima



Background

GEV

Three parameters: **location**, **scale** and **shape**.

$$\Pr\{X \leq z\} = \exp\left\{-\left[1+\boldsymbol{\xi}\left(\frac{z-\boldsymbol{\mu}}{\boldsymbol{\sigma}}\right)\right]^{-1/\boldsymbol{\xi}}\right\}$$

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GEV

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$$\Pr\{X \le z\} = \exp\left\{-\left[1 + \boldsymbol{\xi}\left(\frac{z - \boldsymbol{\mu}}{\boldsymbol{\sigma}}\right)\right]^{-1/\boldsymbol{\xi}}\right\}$$

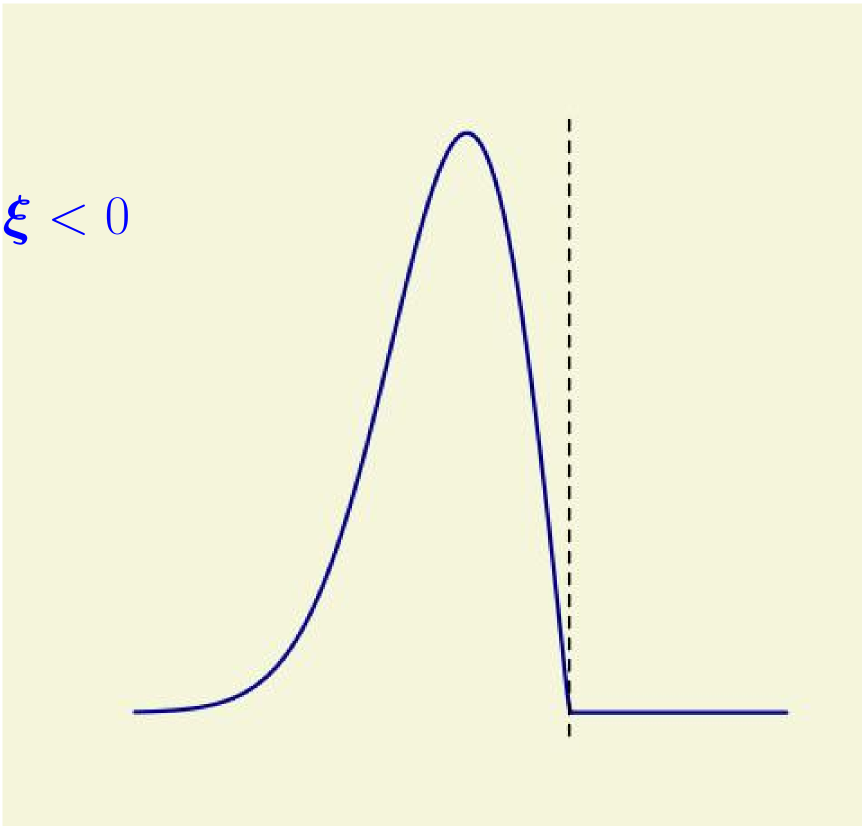
Three types of tail behavior:

1. Bounded upper tail ($\boldsymbol{\xi} < 0$, Weibull),
2. light tail ($\boldsymbol{\xi} = 0$, Gumbel), and
3. heavy tail ($\boldsymbol{\xi} > 0$, Fréchet).

Background

Weibull Type

Bounded upper tail is a function of parameters. Namely, $\mu - \sigma/\xi$.

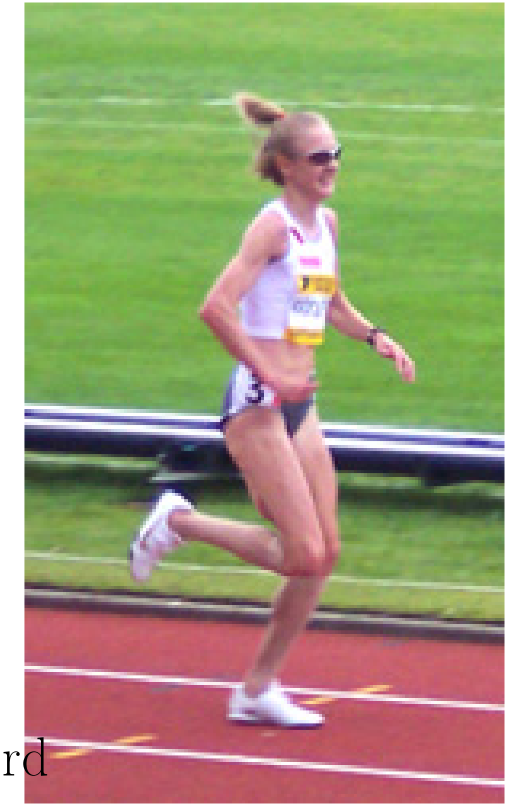


Temperature, Wind Speed, Sea Level

Background

Weibull Type

Predicted Speed Limits for:
Thoroughbreds (Kentucky Derby) $\approx$ 38 mph
Greyhounds (English Derby) $\approx$ 38 mph
Men (100 m distance) $\approx$ 24 mph
Women (100 m distance) $\approx$ 22 mph
Men (10 km distance) $\approx$ 15 mph
Women (marathon distance) $\approx$ 12 mph
Women (marathon distance, population model) $\approx$ 11.45 mph



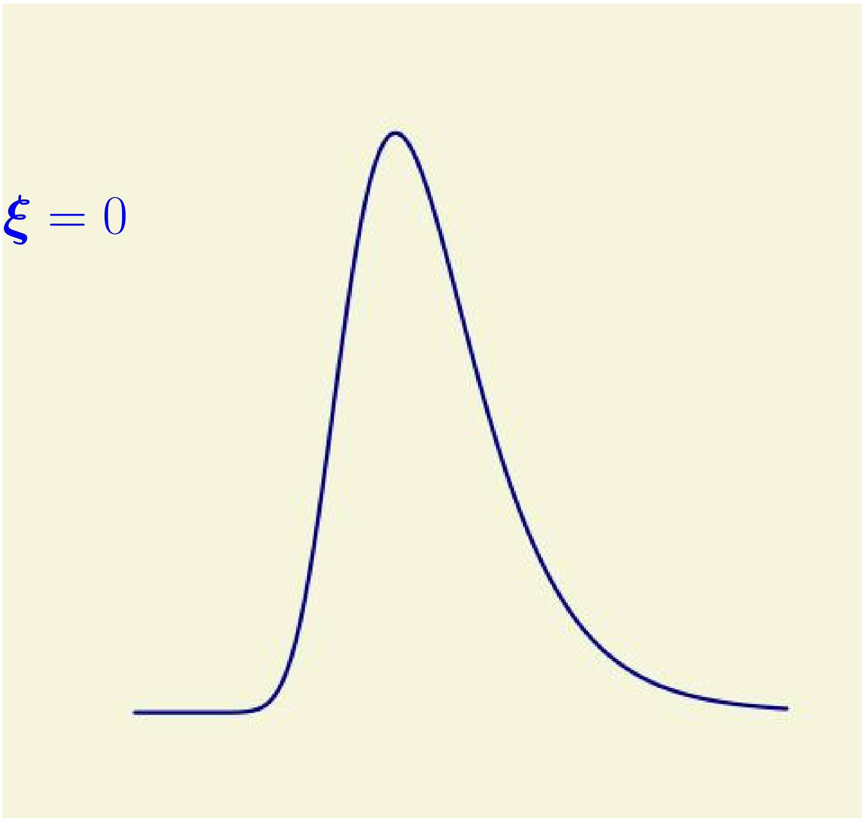
Paula Radcliffe, 11.6 mph world marathon record

Denny, M.W., 2008: Limits to running speed in dogs, horses and humans. *J. Experim. Biol.*, **211**:3836–3849.

Background

Gumbel Type

Light-tailed distribution (i.e., decays exponentially)



Domain of attraction for many common distributions (e.g., the normal distribution)

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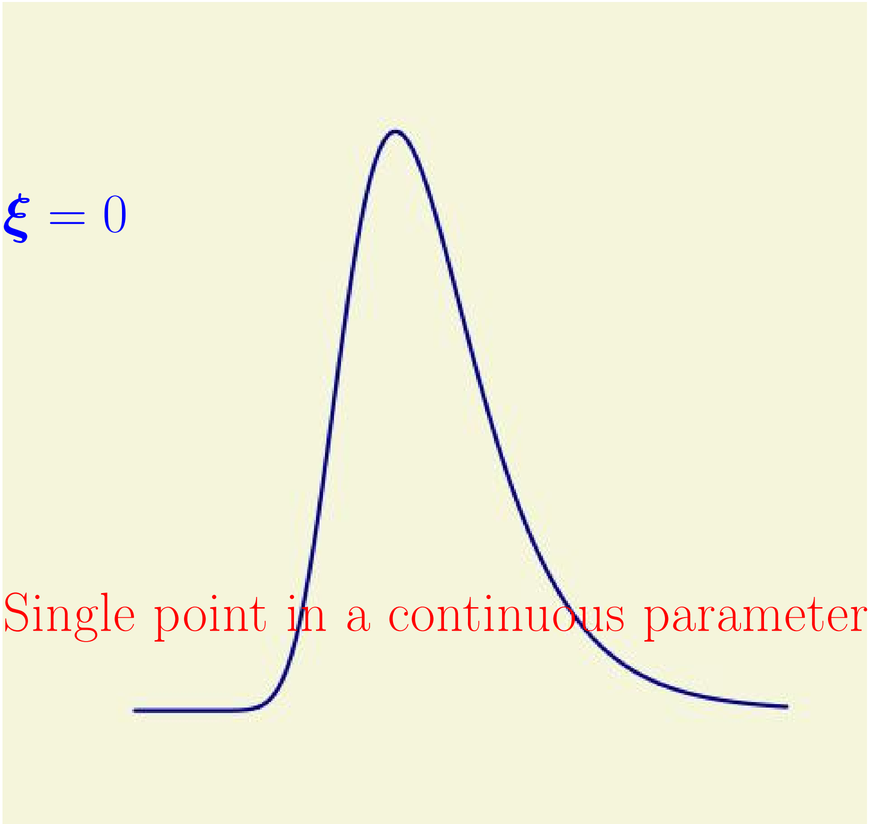


$\boldsymbol{\xi} = 0$

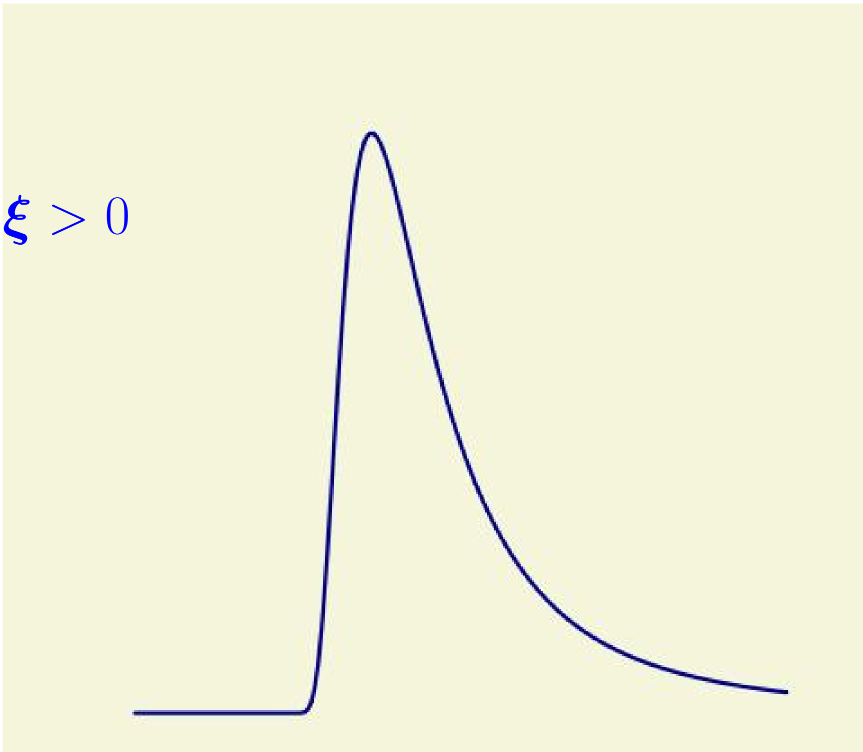
Single point in a continuous parameter space!

Domain of attraction for many common distributions (e.g., the normal distribution)

Background

Fréchet Type

Heavy-tailed distribution (i.e., decays polynomially)
Bounded lower tail at $\boldsymbol{\mu} - \boldsymbol{\sigma}/\boldsymbol{\xi}$.



Precipitation, Stream Flow, Economic Impacts

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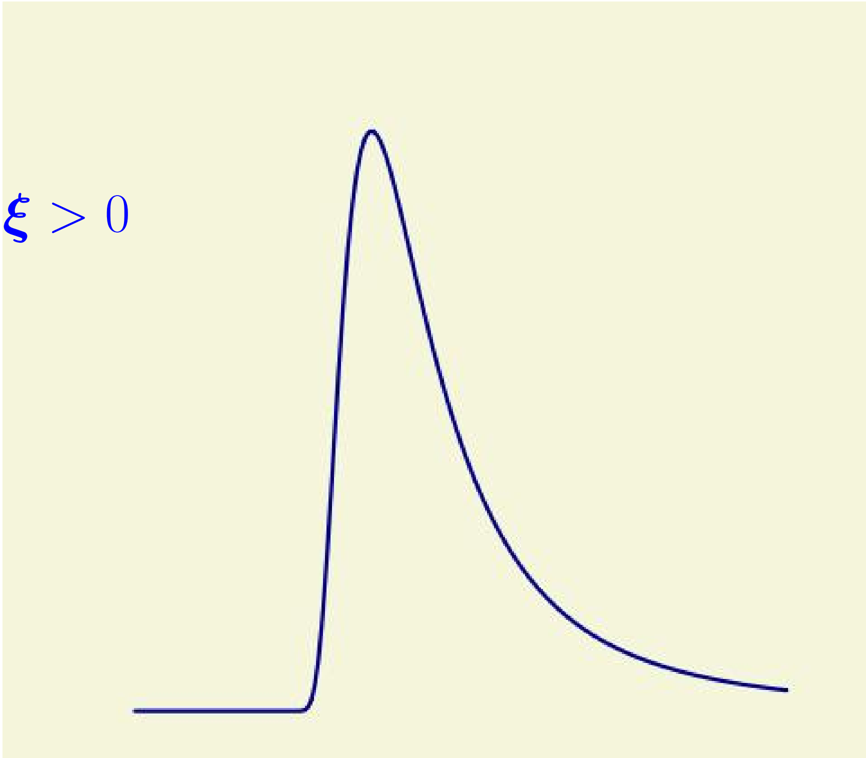
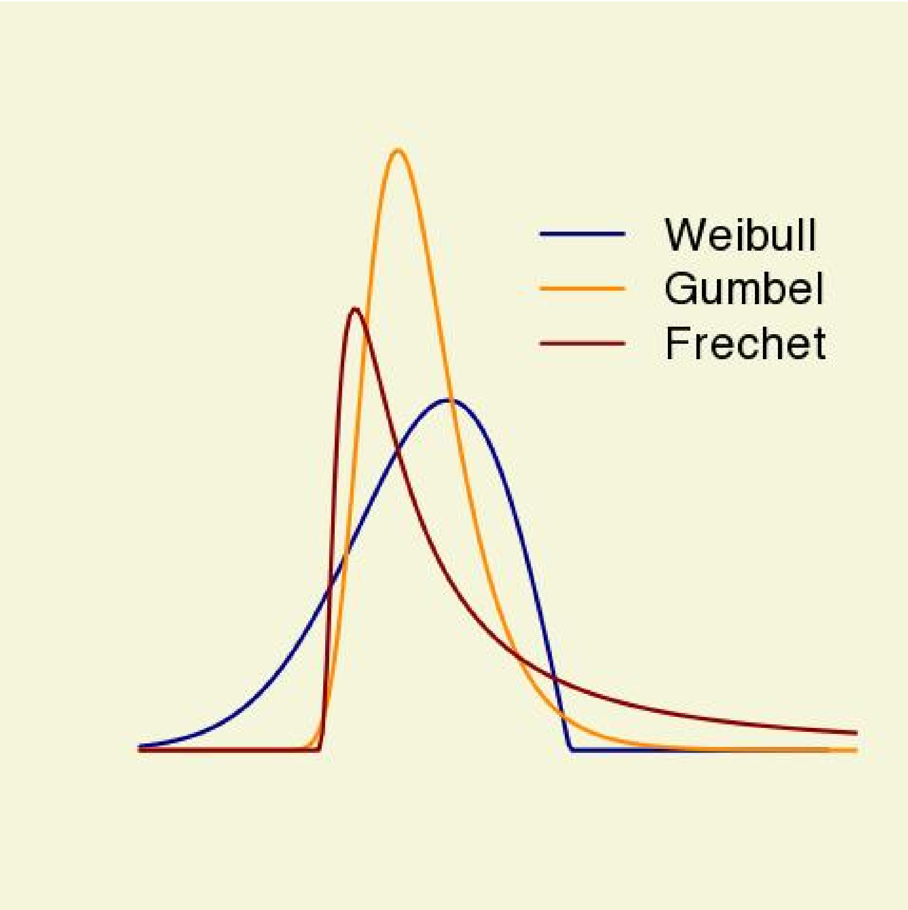


Precipitation, Stream Flow, Economic Impacts

Infinite mean if $\xi \geq 1$!
Infinite variance if $\xi \geq 1/2$!

Background

All three types together



Background

Analogous for Peaks Over a Threshold (POT) approach

Generalized Pareto Distribution (GPD), which has two parameters: **scale** and **shape**. Threshold replaces the **location** parameter.

Three Types:

1. Beta ($\boldsymbol{\xi} < 0$), bounded above at threshold$-\boldsymbol{\sigma}/\boldsymbol{\xi}$
2. Exponential ($\boldsymbol{\xi} = 0$), light tail
3. Pareto ($\boldsymbol{\xi} > 0$), heavy tail

Background

Minima

Same as maxima using the relation:

$$\min\{X_1, \ldots, X_n\} = -\max\{-X_1, \ldots, -X_n\}$$

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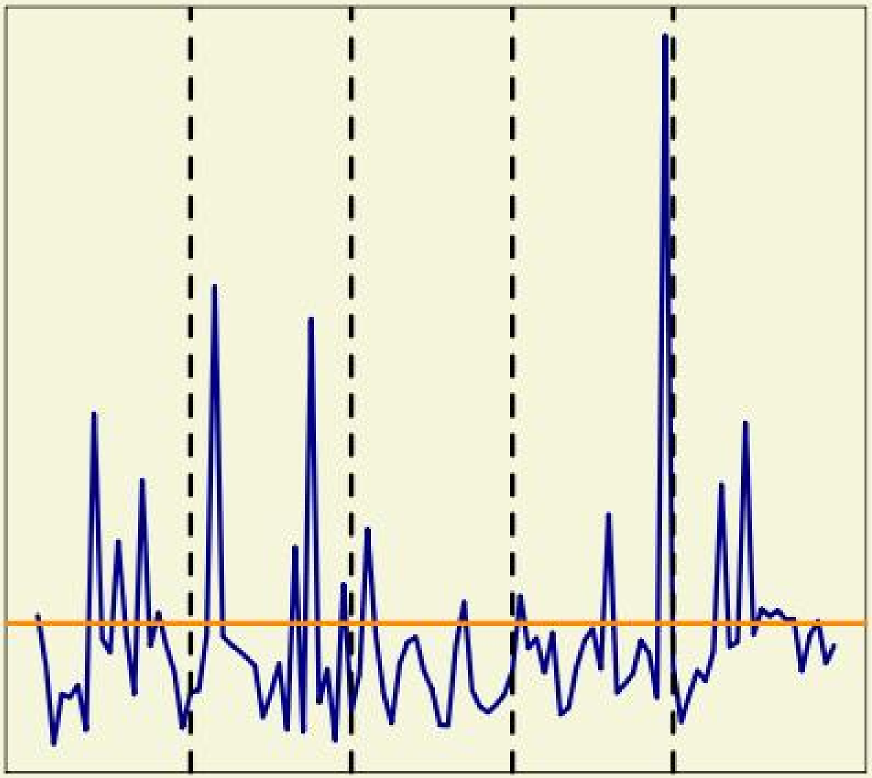
Analogous for POT approach: Look at negatives of deficits under a threshold instead of excesses over a threshold.

Background

Block Maxima vs. POT



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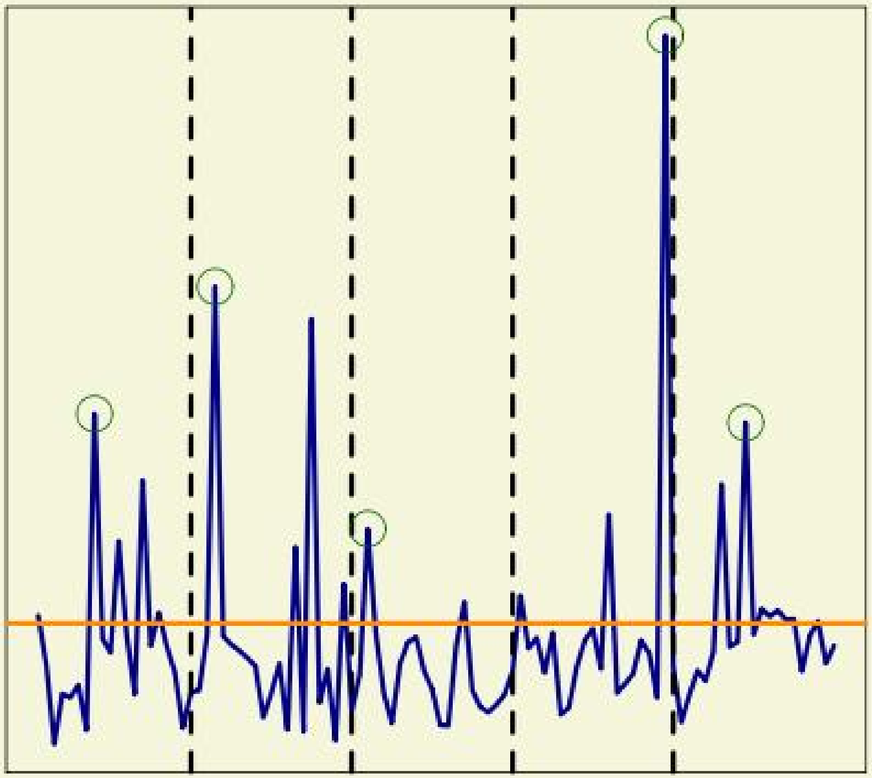

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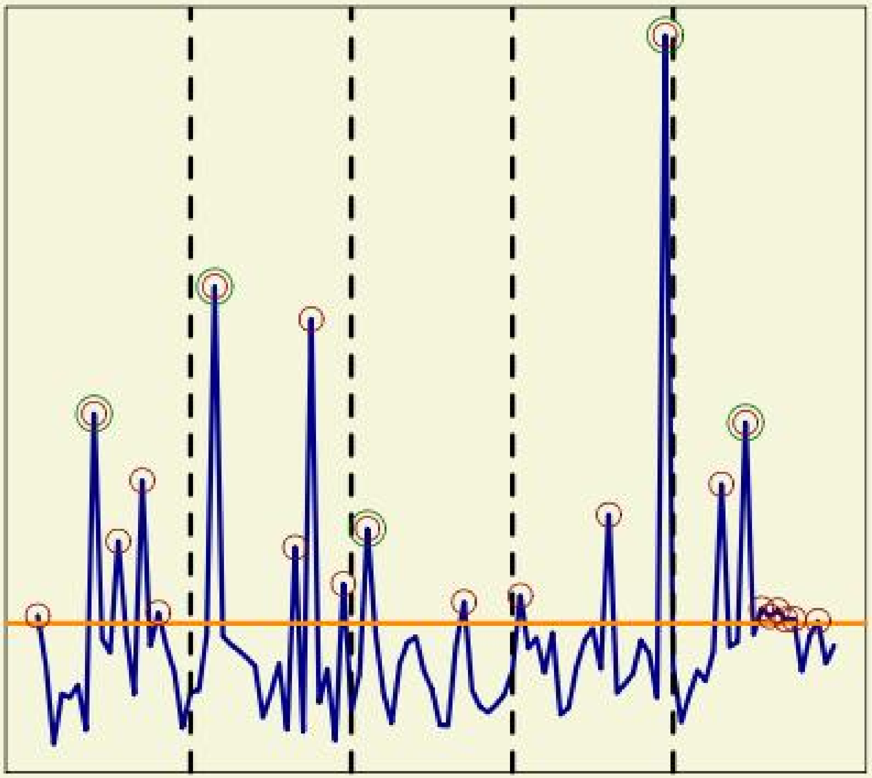


Examples

Fort Collins, Colorado

daily precipitation amount

- Time series of daily precipitation amount (inches), 1900–1999.
- Semi-arid region.
- Marked annual cycle in precipitation (wettest in late spring/early summer, driest in winter).
- No obvious long-term trend.
- Recent flood, 28 July 1997. (substantial damage to Colorado State University)

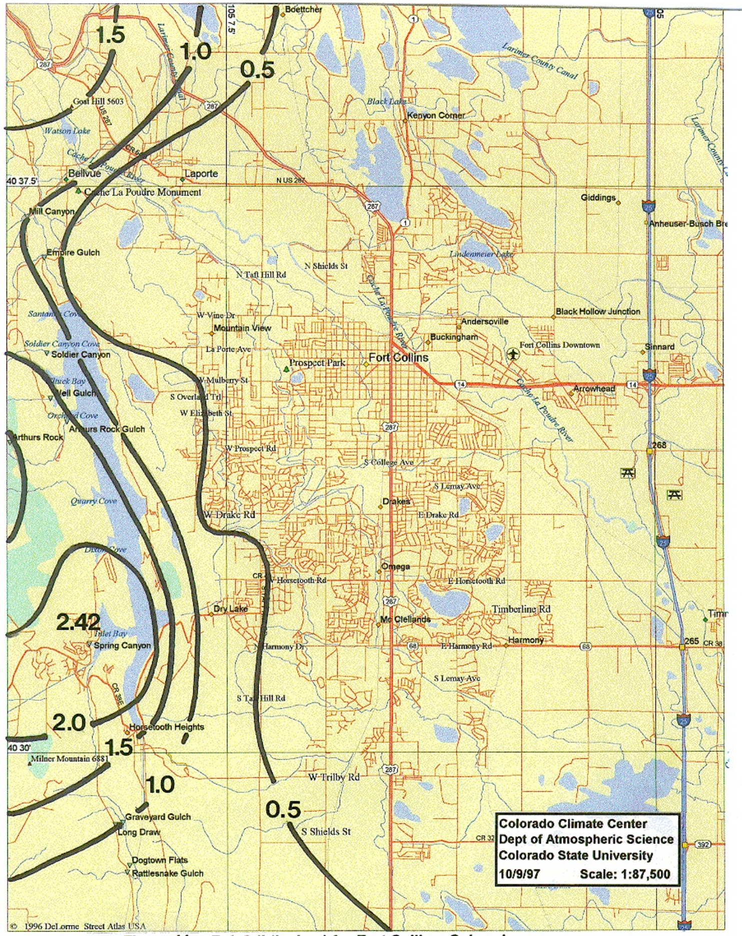
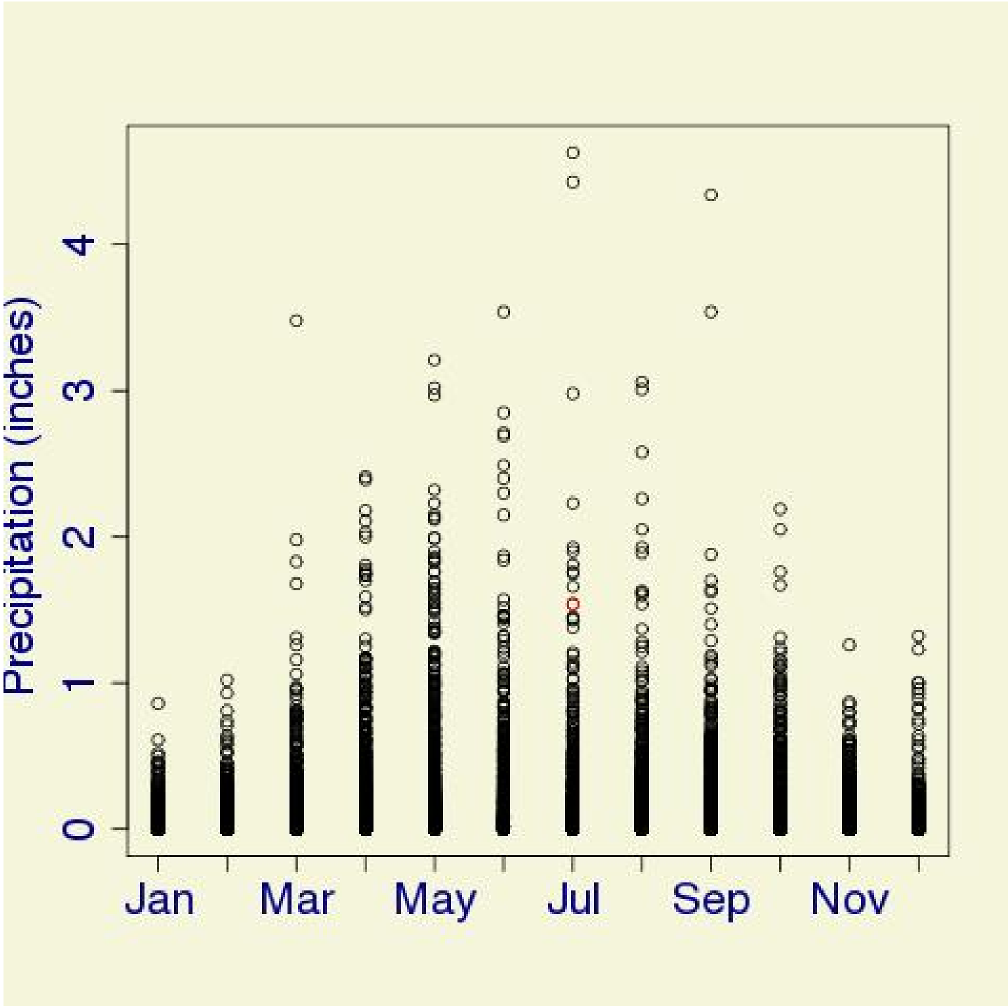


Figure 11. Rainfall (inches) for Fort Collins, Colorado, for 4:00-9:00 p.m. MDT for July 27, 1997

http://ccc.atmos.colostate.edu/~odie/rain.html

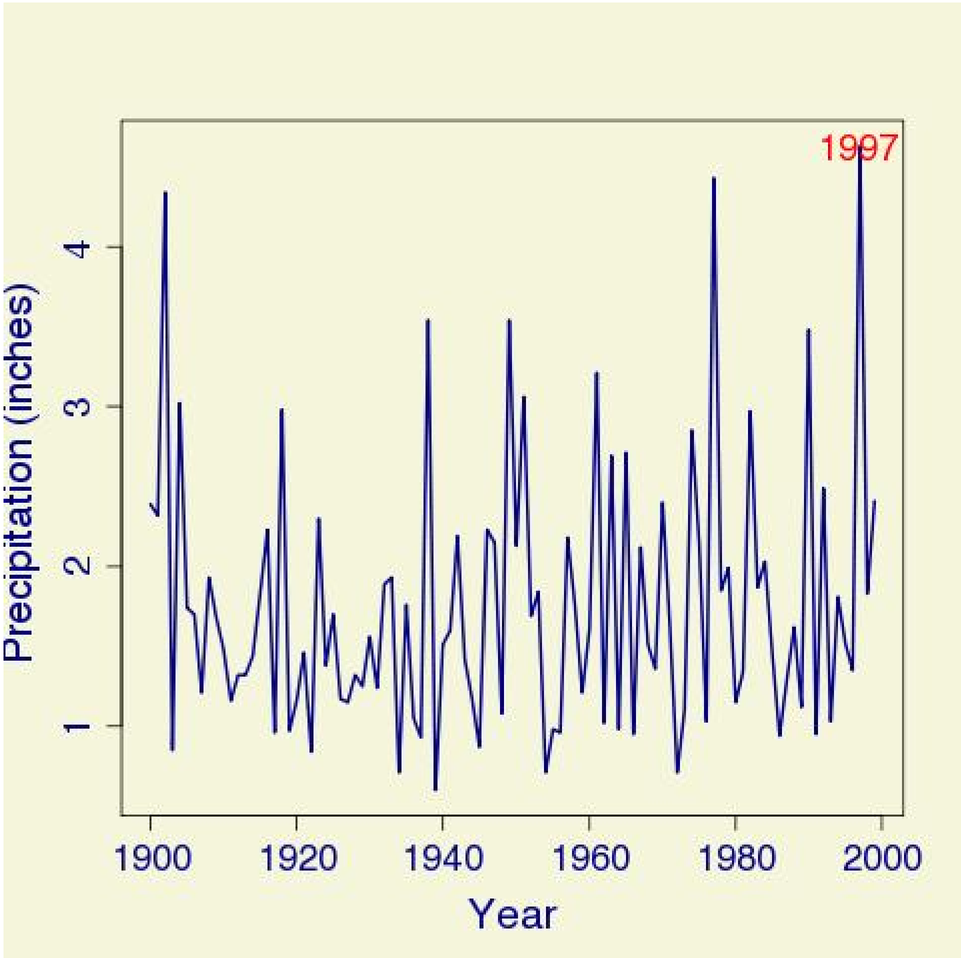
Examples

Fort Collins, Colorado precipitation



Examples

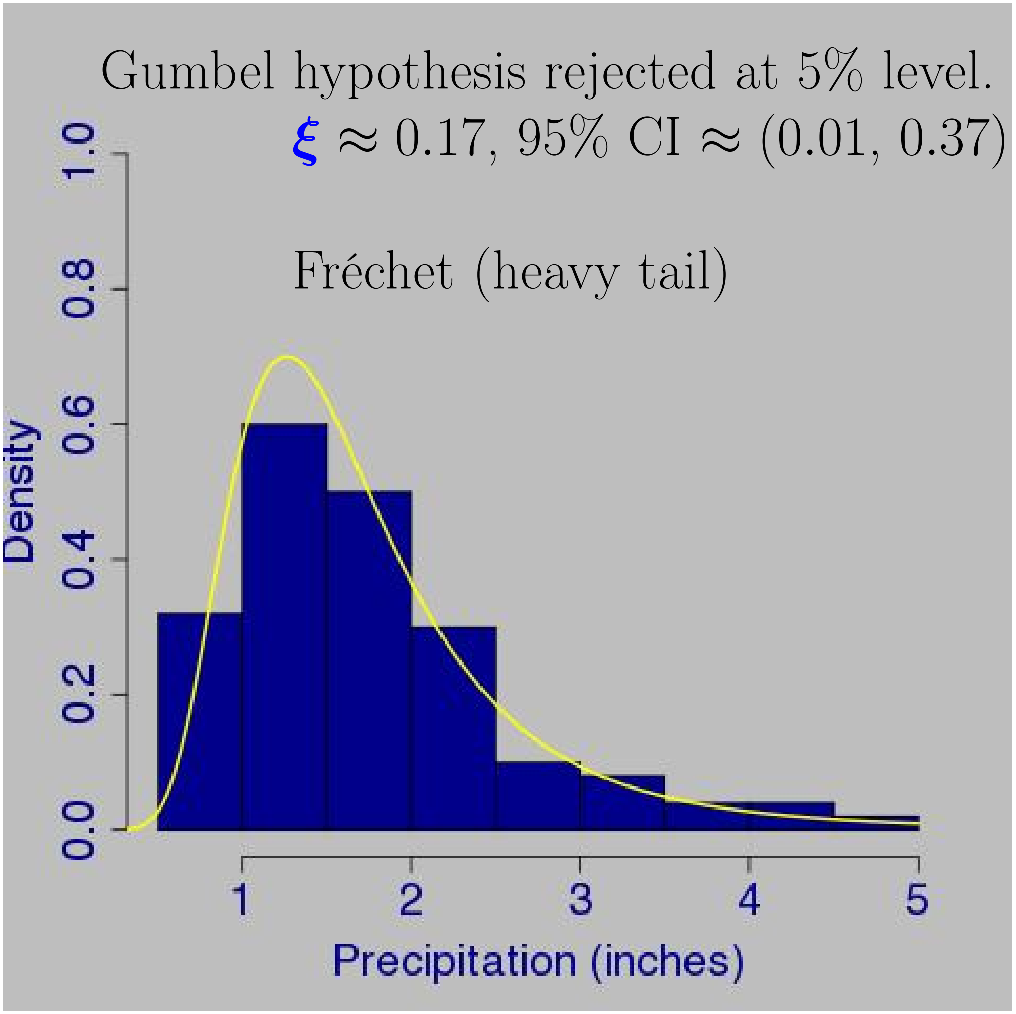
Fort Collins, Colorado Annual Maximum Precipitation



How often is such an extreme expected?

Examples

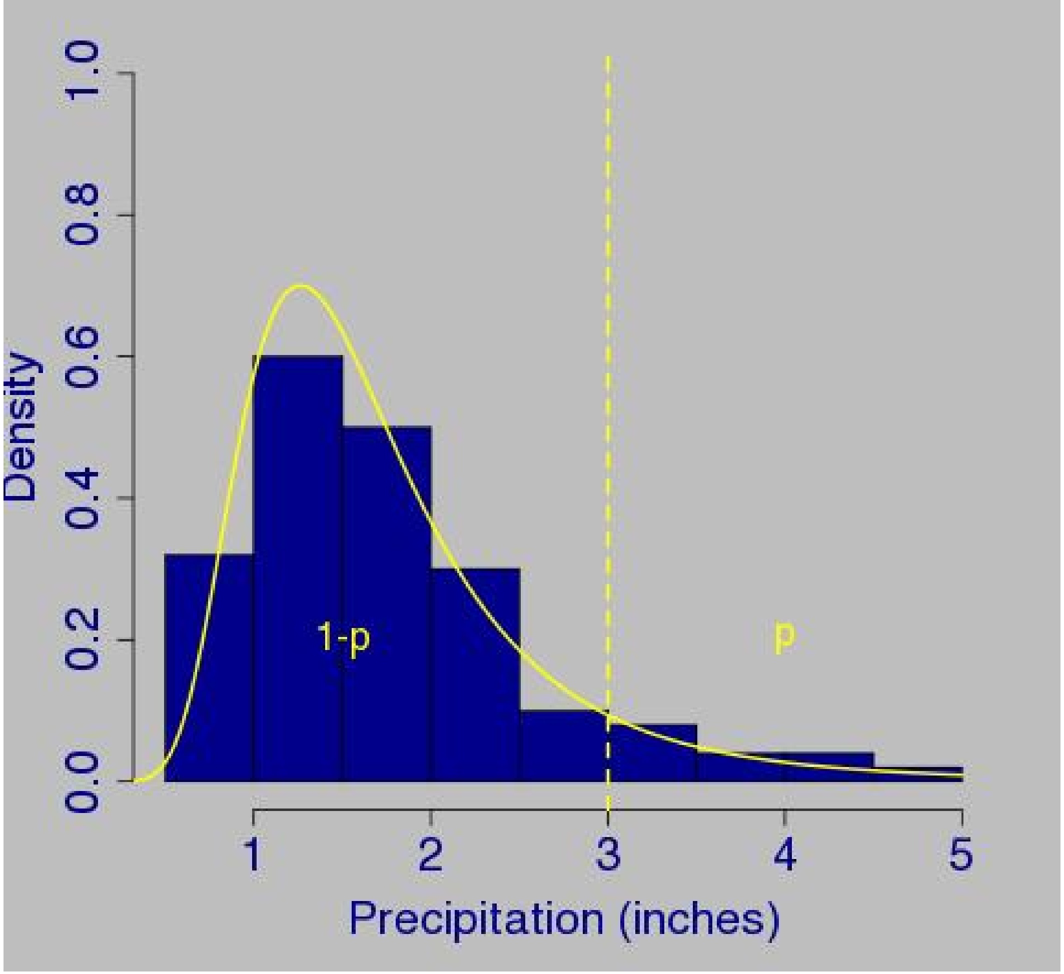
Fort Collins, Colorado precipitation



Examples

Fort Collins, Colorado precipitation

Risk Communication

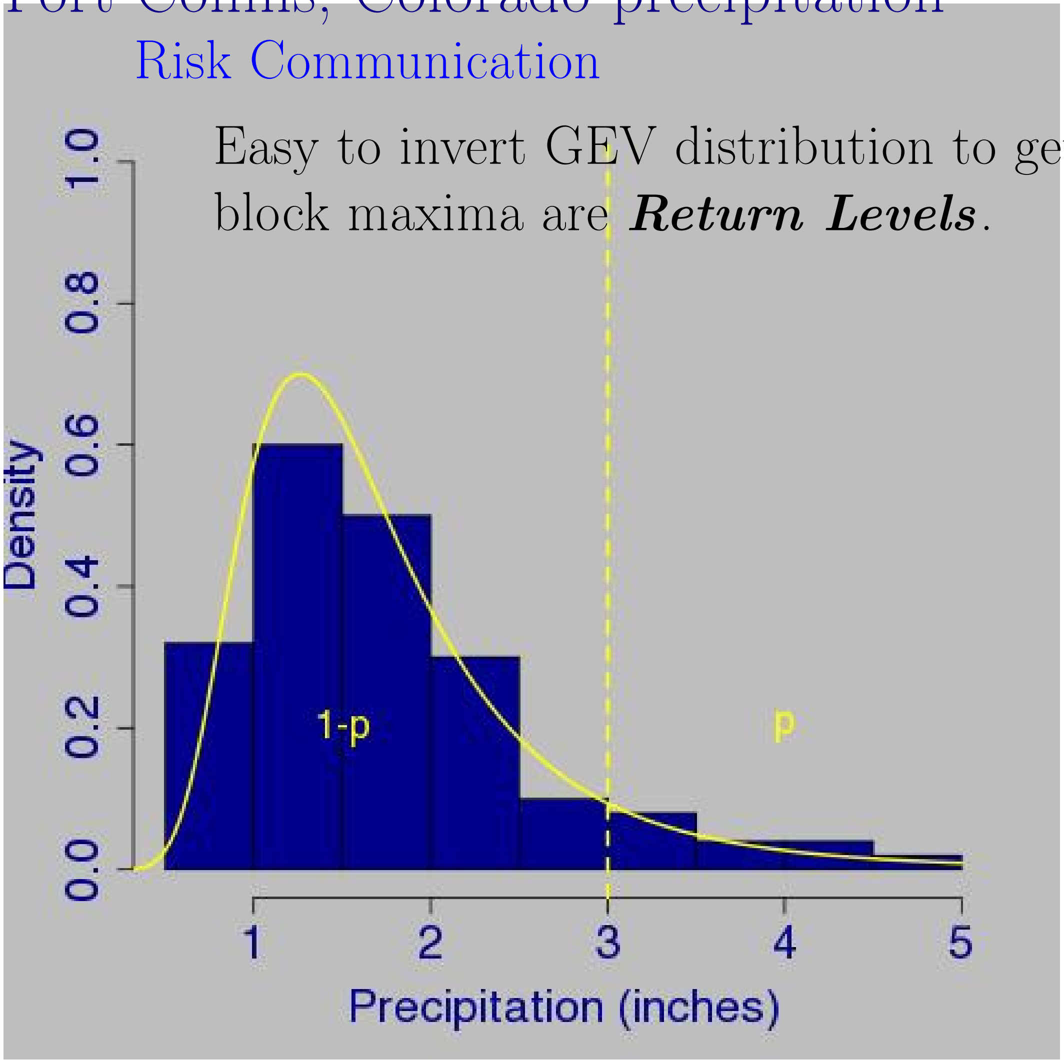


Examples

Fort Collins, Colorado precipitation

Risk Communication

Easy to invert GEV distribution to get quantiles, which for block maxima are ***Return Levels***.



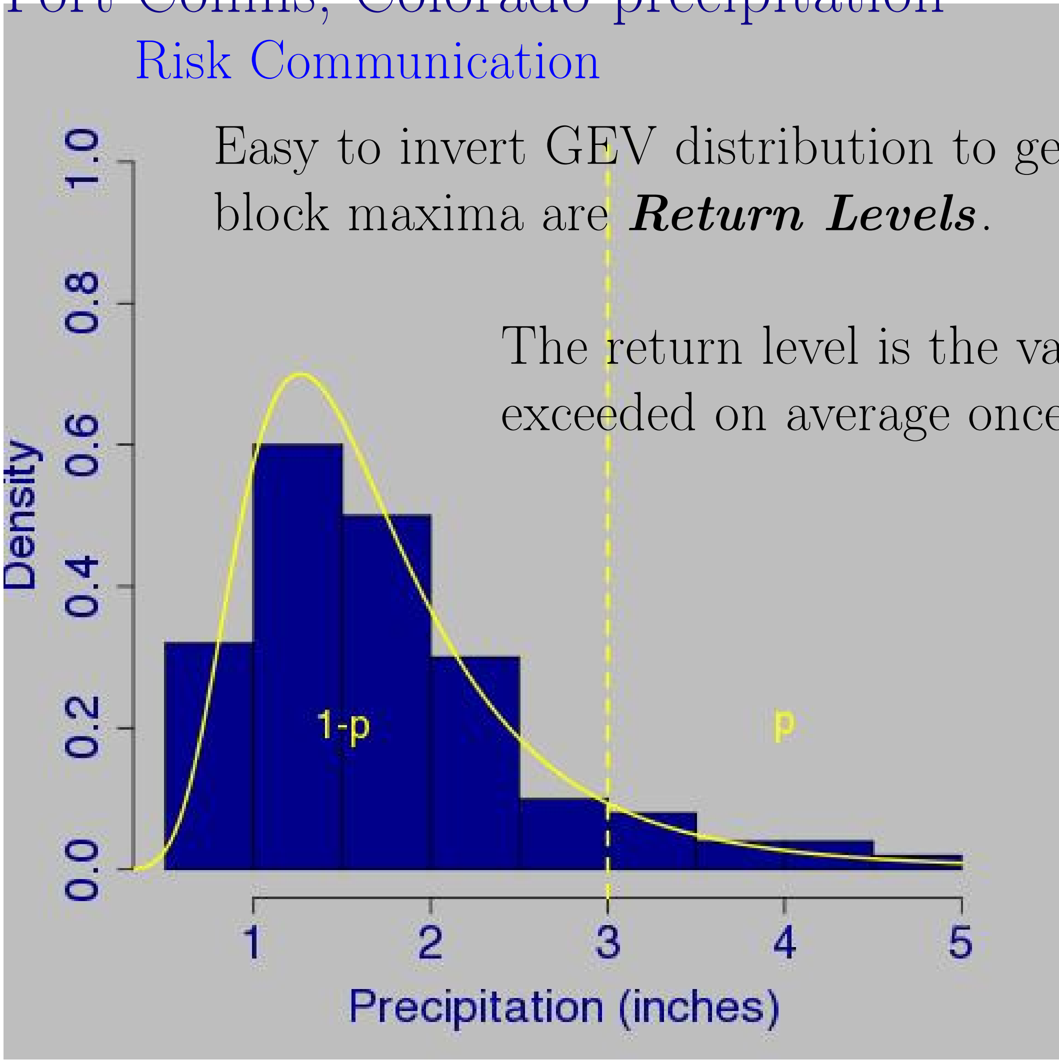
Examples

Fort Collins, Colorado precipitation

Risk Communication

Easy to invert GEV distribution to get quantiles, which for block maxima are ***Return Levels***.

The return level is the value expected to be exceeded on average once every 1/p years.



Examples

Fort Collins, Colorado precipitation

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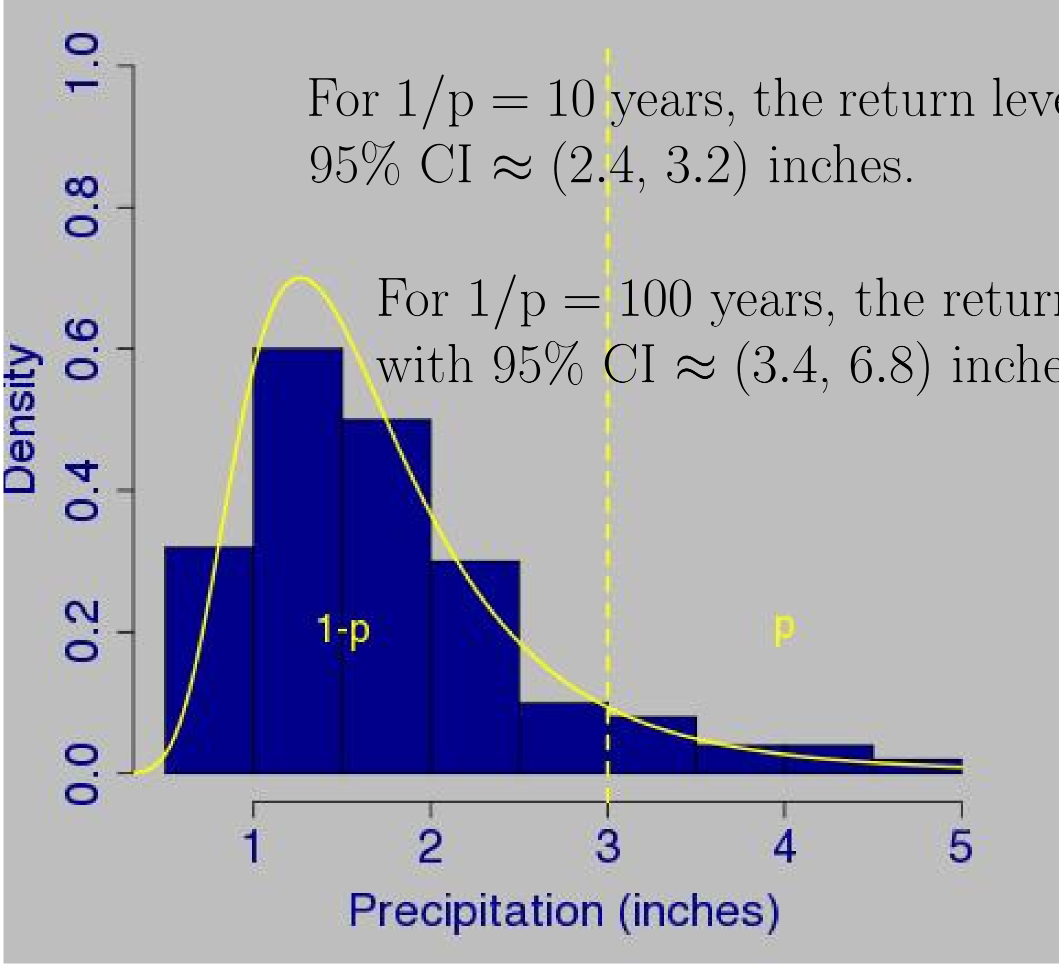

For $1/p = 10$ years, the return level is ≈ 2.8 inches with 95% CI $\approx (2.4, 3.2)$ inches.

For $1/p = 100$ years, the return level is ≈ 5.1 inches with 95% CI $\approx (3.4, 6.8)$ inches.

Examples

Fort Collins, Colorado precipitation

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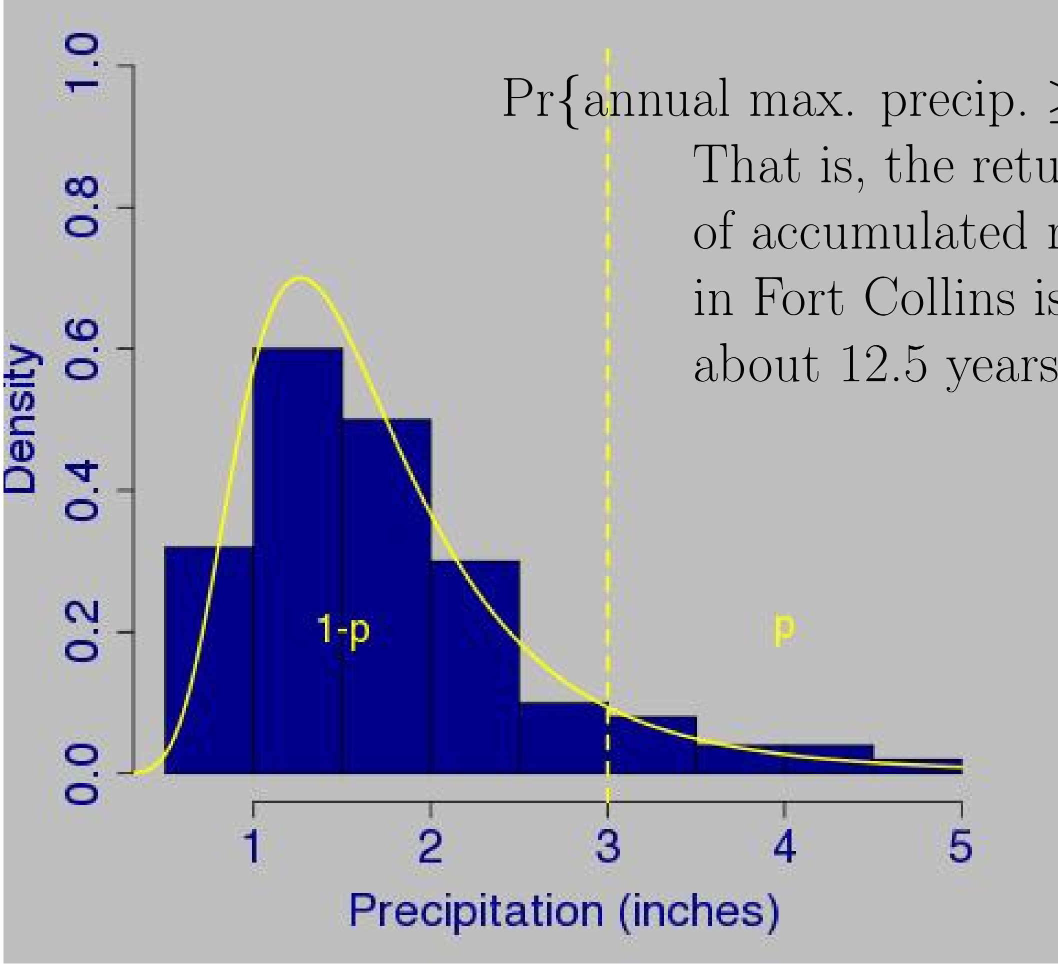


Pr{annual max. precip. $\geq$ 3 inches} $\approx$ 0.08

That is, the return period for 3 inches of accumulated rainfall at this gauge in Fort Collins is estimated to be about 12.5 years.

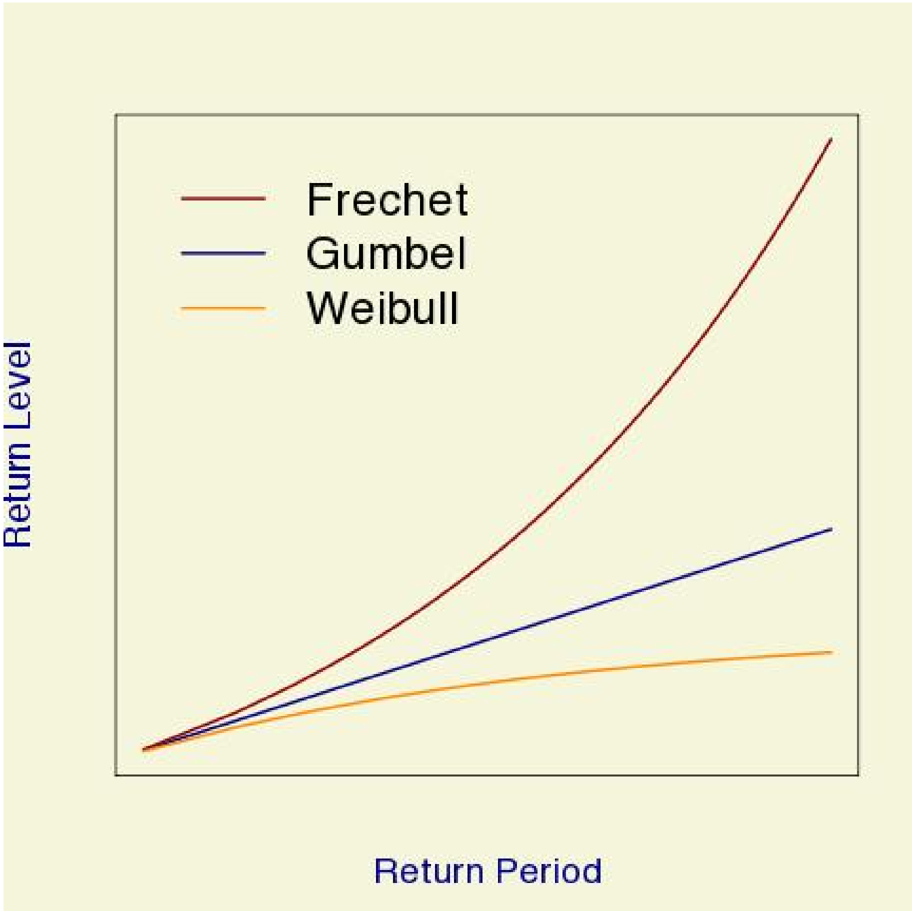
Examples

Fort Collins, Colorado precipitation

Can also obtain other information, such as

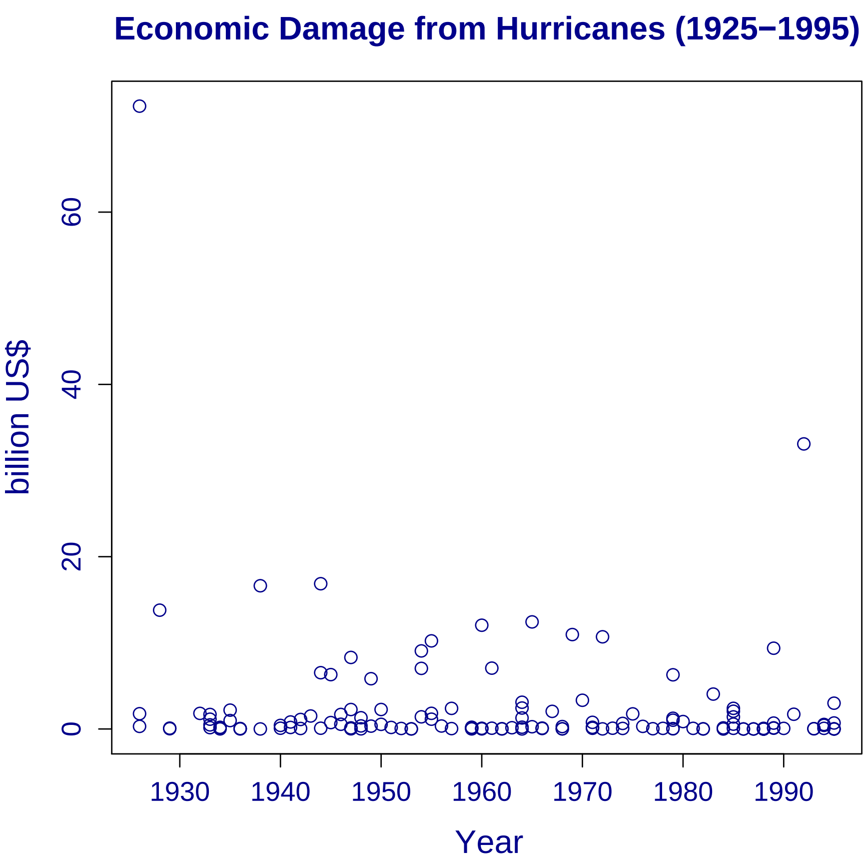
- Mean annual maximum daily precipitation accumulation ≈ 1.76 inches ($\neq \hat{\boldsymbol{\mu}} \approx 1.35$).
- Variance is ≈ 0.84 inches^2.
- Standard deviation is ≈ 0.92 inches ($\neq \hat{\boldsymbol{\sigma}} \approx 0.53$).

Background



Examples

Hurricane damage



Economic damage caused by hurricanes from 1926 to 1995.

Trends in societal vulnerability removed.

Excess over threshold of $u = 6$ billion US$.

Examples

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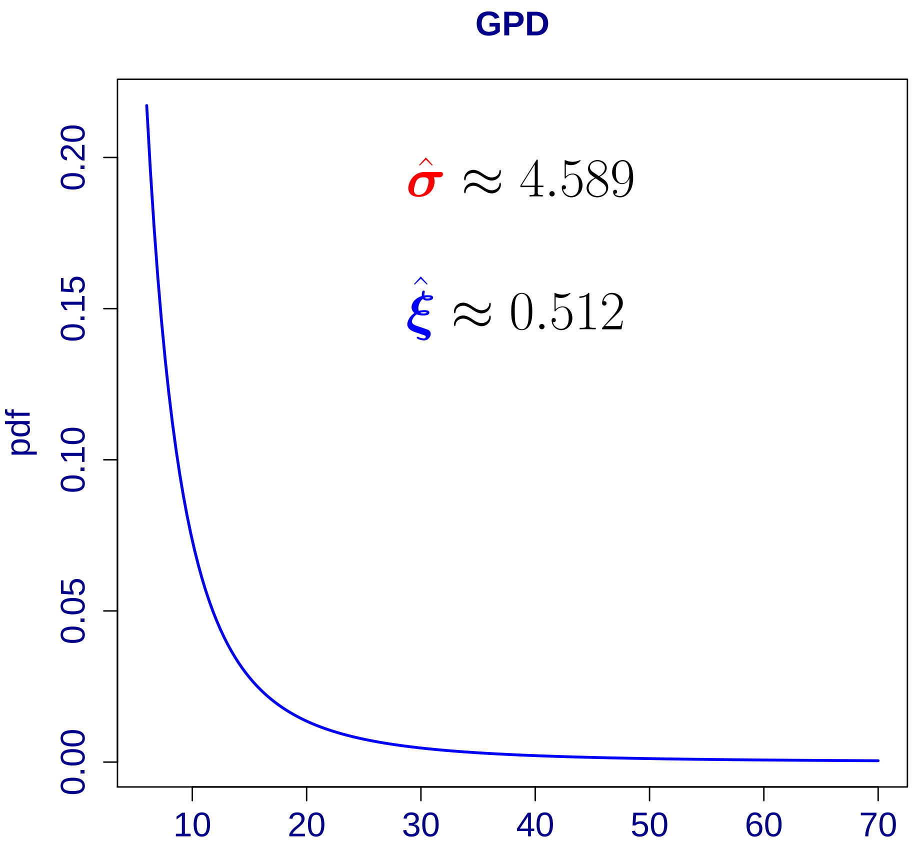


95% CI for shape parameter using profile likelihood.
$\approx (0.05, 1.56)$
Heavy tail!

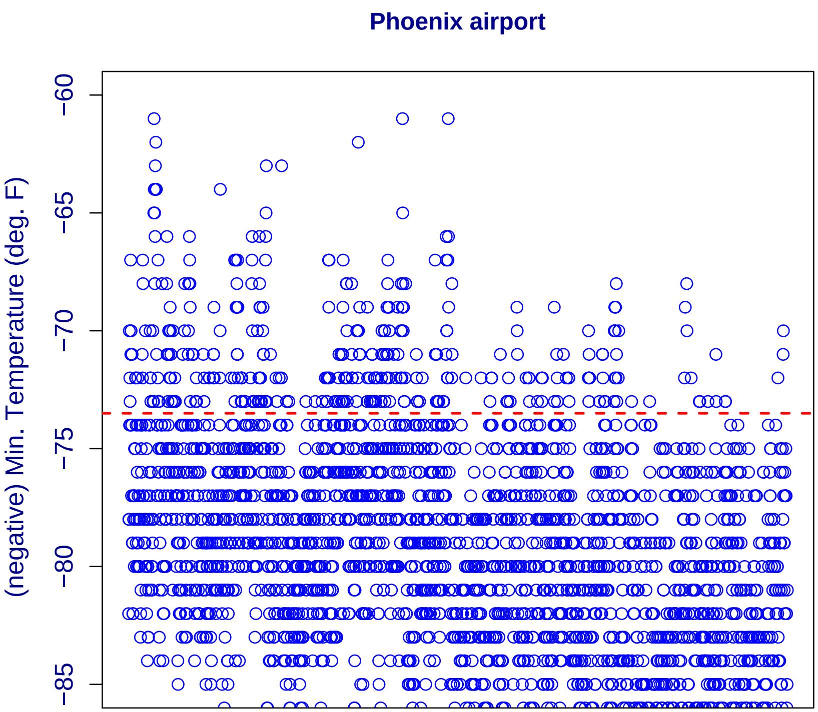
Examples

Hurricane Dennis (2005)
Caused at least 89 deaths and
2.23 billion USD in damage.

Impactful despite being under the 6 billion USD threshold!

Examples

Phoenix (airport) minimum temperature (^{o}F)



July and August 1948–1990.

Urban heat island (warming trend as cities grow).

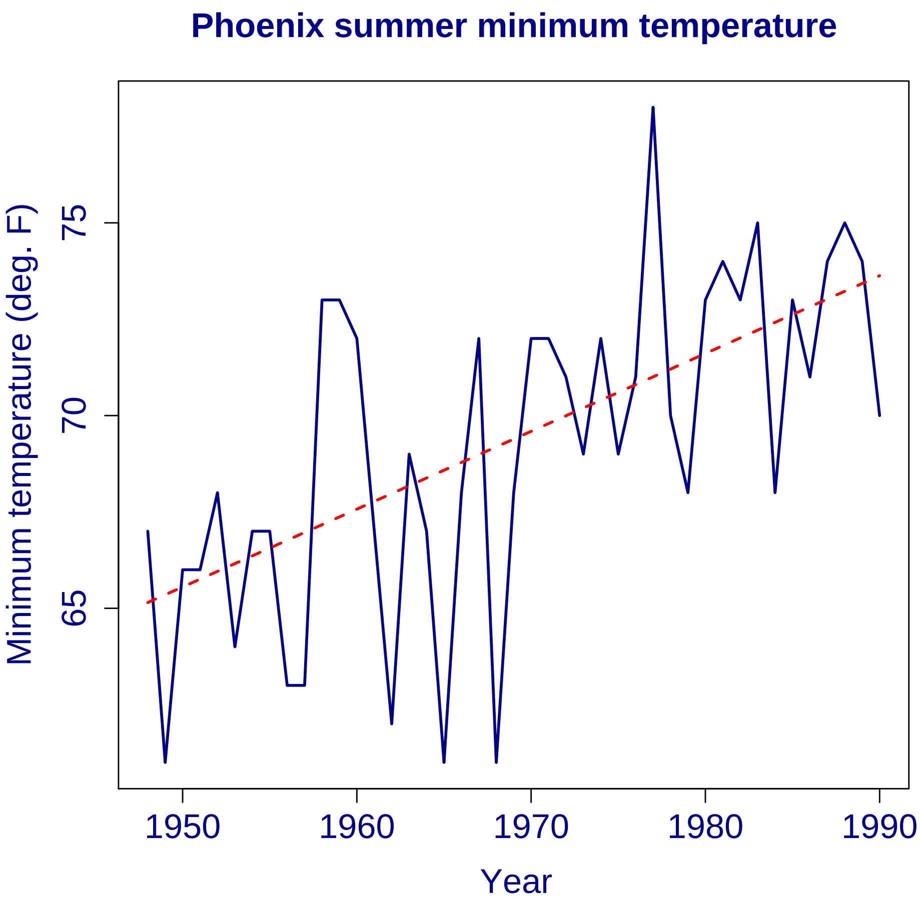
Model lower tail as upper tail after negation.

Dependence over the threshold.

Temporal trend!

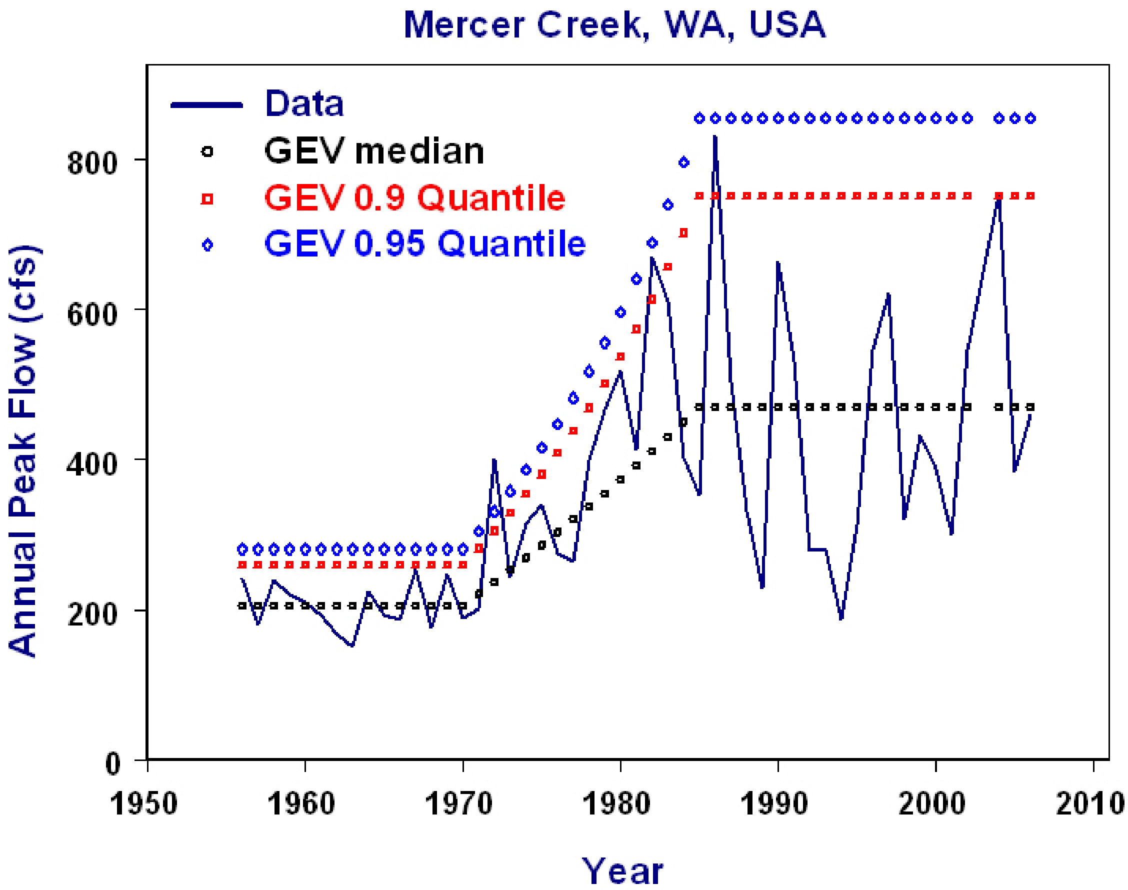
Examples

Phoenix minimum temperature



Regression-like approach. Covariate information in GEV parameters.

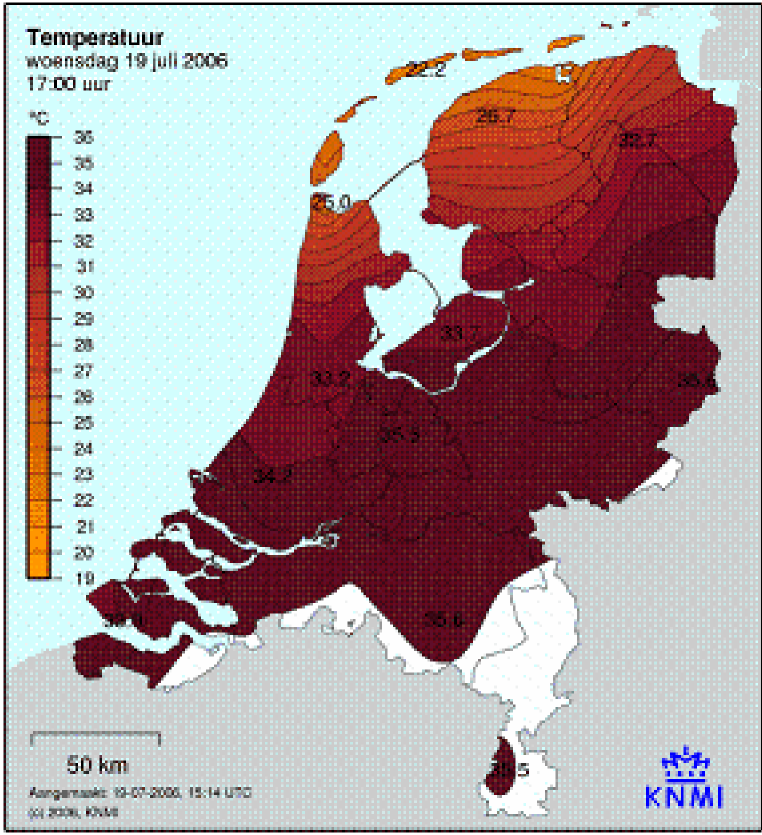
Examples



Rapid urban development started about 1970.

http://pubs.usgs.gov/fs/FS-229-96

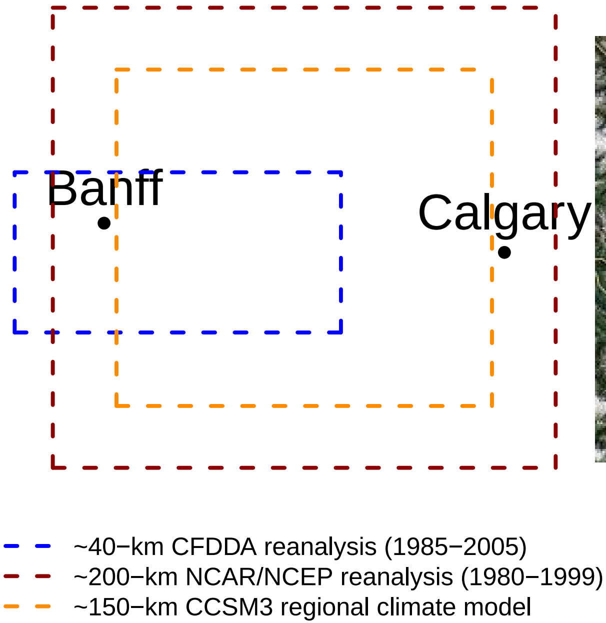
Extreme Value Problems in Climatology

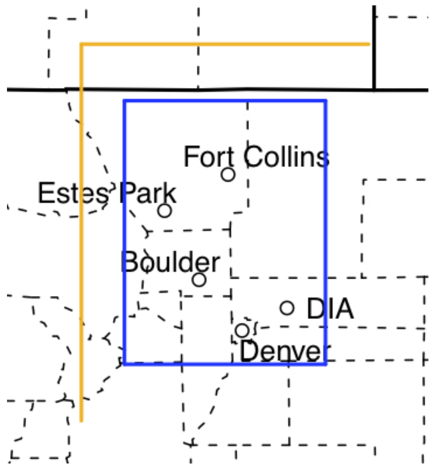


2006 European Heat Wave
(Fig. from KNMI)

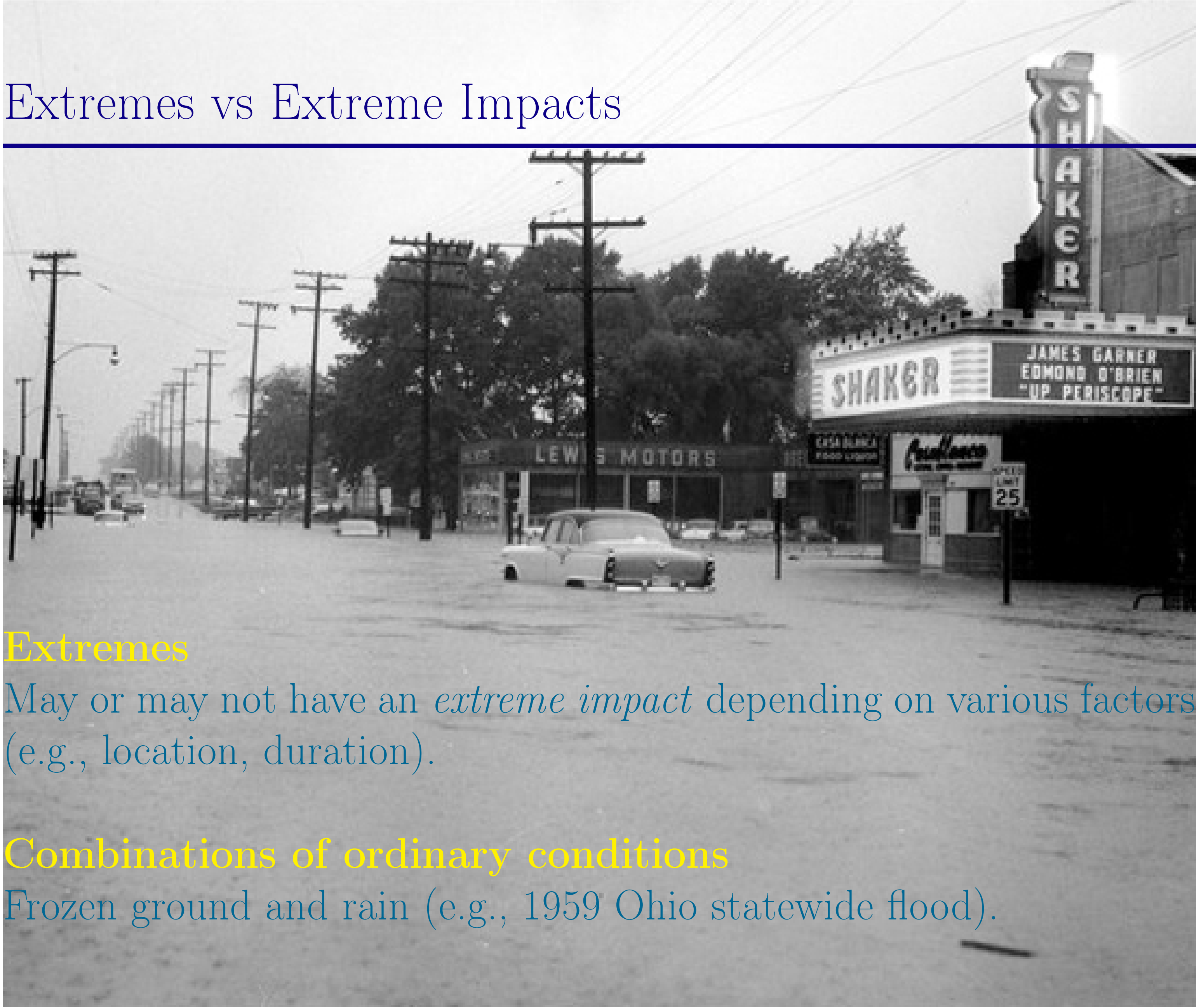
F5 Tornado in Elie Manitoba
on Friday, June 22nd, 2007

Extreme Value Problems in Climatology





Extremes vs Extreme Impacts



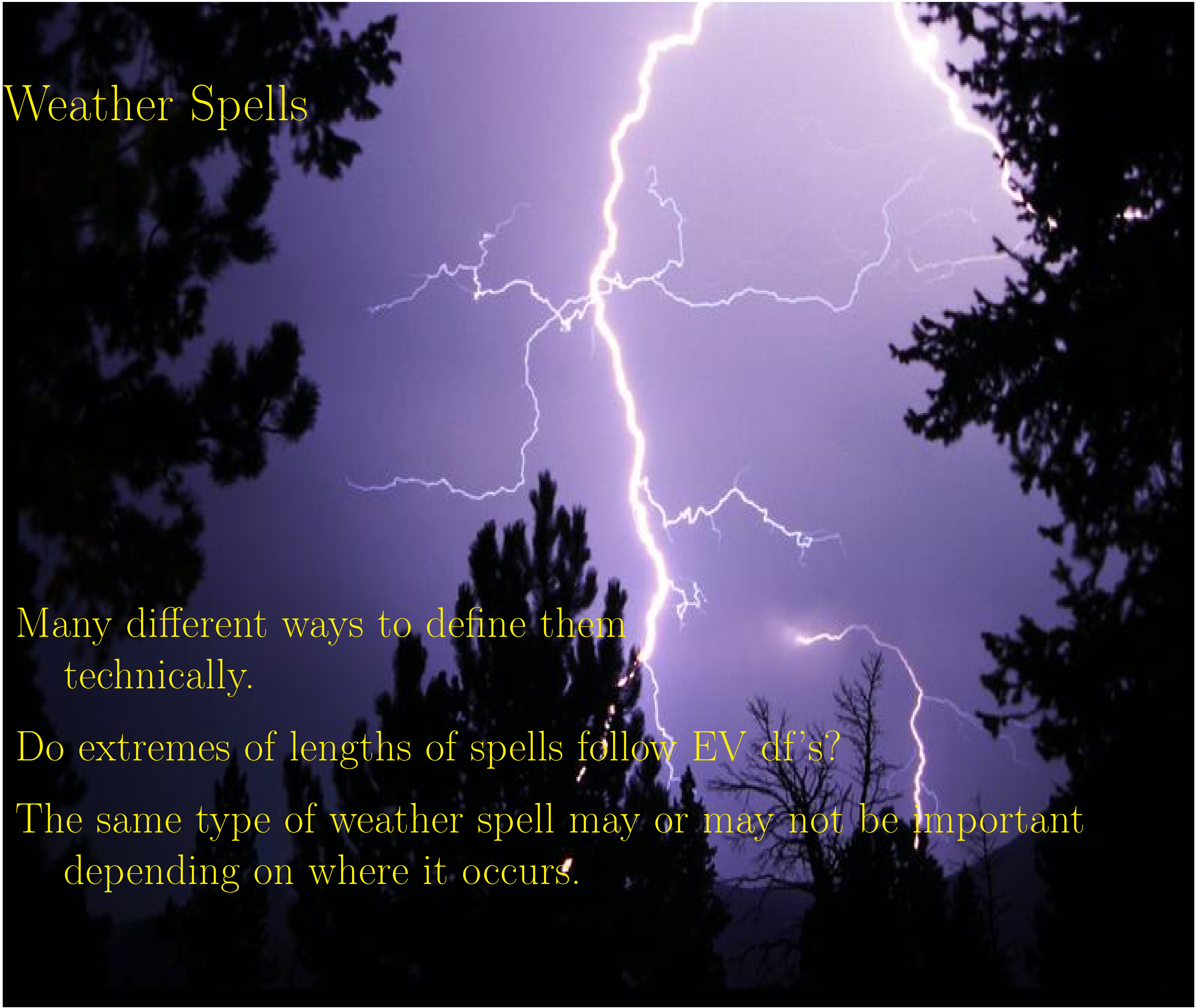
Extremes

May or may not have an *extreme impact* depending on various factors (e.g., location, duration).

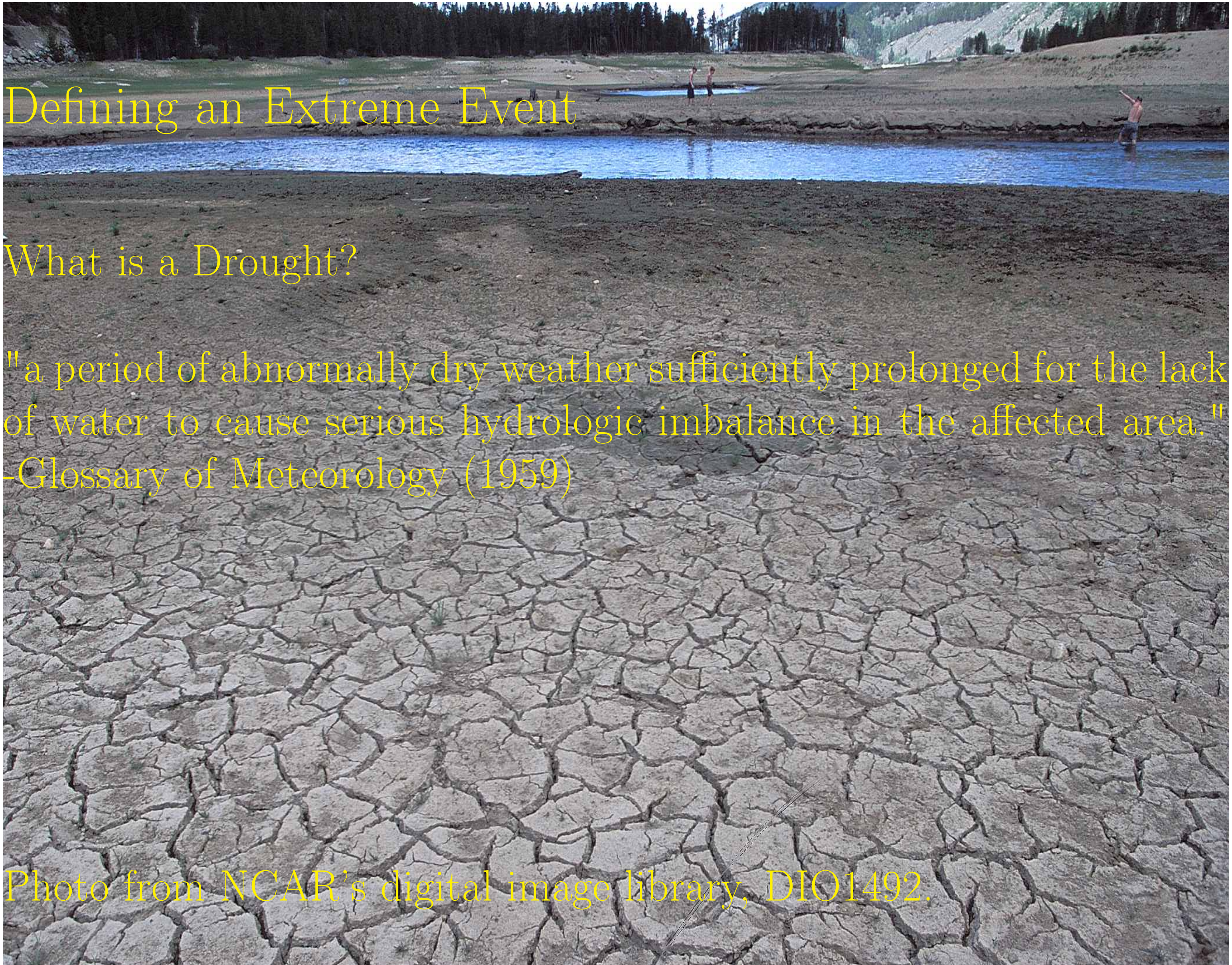
Combinations of ordinary conditions

Frozen ground and rain (e.g., 1959 Ohio statewide flood).

Weather Spells



- Many different ways to define them technically.
- Do extremes of lengths of spells follow EV df's?
- The same type of weather spell may or may not be important depending on where it occurs.



Defining an Extreme Event

What is a Drought?

"a period of abnormally dry weather sufficiently prolonged for the lack of water to cause serious hydrologic imbalance in the affected area."
-Glossary of Meteorology (1959)

Photo from NCAR's digital image library, DIO1492.

Defining an Extreme Event

What is a Drought?

Meteorological–a measure of departure of precipitation from normal. Due to climatic differences, what might be considered a drought in one location of the country may not be a drought in another location.

Agricultural–refers to a situation where the amount of moisture in the soil no longer meets the needs of a particular crop.

Hydrological–occurs when surface and subsurface water supplies are below normal.

Socioeconomic–refers to the situation that occurs when physical water shortages begin to affect people.

http://www.wrh.noaa.gov/fgz/science/drought.php?wfo=fgz

Defining an Extreme Event

What is a Heat Wave?

(e.g., Meehl and Tebaldi, 2004, Science, **305**, 994–997):

- Three-day worst heat event: mean annual 3-day warmest nighttime minima event.
- Threshold excess: The longest period of consecutive days satisfying:

  1. daily maximum temperature above T1 for at least three days,
  2. average daily maximum temperature above T1 for entire period, and
  3. daily maximum temperature above T2 for every day of entire period.

T1 = 97.5th percentile of the df of maximum temperatures in observed and present-day climate simulations. T2 = 81st percentile.

Weather Spells

Some things to consider

- How should a spell be defined?
  - In terms of impacts? (Varies greatly by region)
  - In terms of perceived impact (e.g., perceived temperature)? (Varies by person)
  - By combinations of variables? (not necessarily extreme)
  - Duration of some persistent event?
  - Can/Should EVD's be used for these types of phenomena?
- Often only seasons are examined (e.g., summer for heat waves), but times of seasons may be changing, and spells may also shift in time.
- Large-scale phenomena important, as well as local conditions and characteristics.

Severe Weather

As climate models become better in resolution, they may resolve some severe weather phenomena, such as hurricanes. However, other types of severe weather may still require higher resolution.

- Use large-scale indicators to analyze conditions ripe for severe weather,
- Use climate models as drivers for finer scale weather models,
- Statistical approach to current trends in observations,
- Model EVD with means and variances as covariates,
- Other?

Extreme Value Problems in Climatology: Discussion

- How should extreme events be defined? Deadliness? Perception-based? Statistically? Economically? Other?
- What is the relationship between changes in the mean and changes in extremes? What about variability? Higher order moments?
- If climate models project the distribution of atmospheric variables, then do they accurately portray them? Enough so that extrema are correctly characterized?
- If climate models only project the mean, then can anything be said about extremes?
- How can it be determined if small changes in high values of large-scale indicators lead to a shift in the distribution of severe weather conditional on the indicators?

Extreme Value Problems in Climatology: Discussion

- How do we verify climate models, especially for inferring about extremes?
- Extremes are often largely dependent on local conditions (e.g., topography, surface conditions, atmospheric phenomena, etc.), as well as larger scale processes.
- Can a *metric* for climate change pertaining to extremes be developed that makes sense, and provides reasonably accurate information?
- How can uncertainty be characterized? Is there too much uncertainty to make inferences about extremes?
- How can spatial structure be taken into account for extremes?
- Many extreme events, and especially extreme impact events, result from multivariate processes. How can this be addressed?

Economic Crisis

Salmon, F., 2009: Recipe for disaster: The formula that killed Wall Street, *Wired magazine*, 23 February, 2009, 7 pp., Available at: http://www.wired.com/print/techbiz/it/magazine/17-03/wp_quant

Embrechts, P., Lectures on, "Did a Mathematical Formula Really Blow Up Wall Street?"

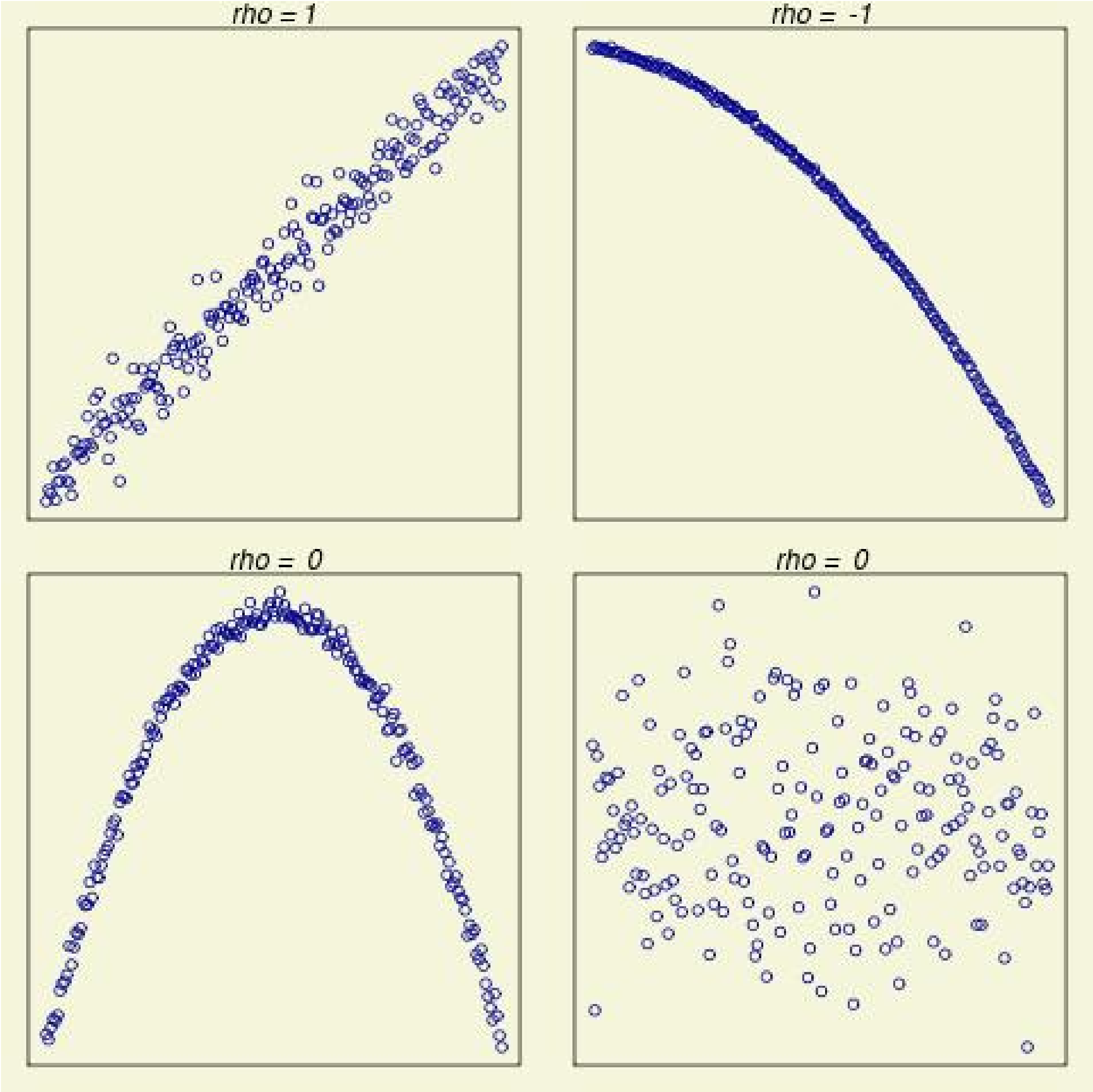
$$\Pr[T_A < 1, T_B < 1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$$

Gaussian copula (aka, David X. Li's formula)

"His method was adopted by everybody from bond investors and Wall Street banks to ratings agencies and regulators. And it became so deeply entrenched–and was making people so much money–that warnings about its limitations were largely ignored." –Salmon, 2009.

Economic Crisis

Correlation



Slide from Paul Embrechts

Professor, ETH Zurich, Dept. of Mathematics

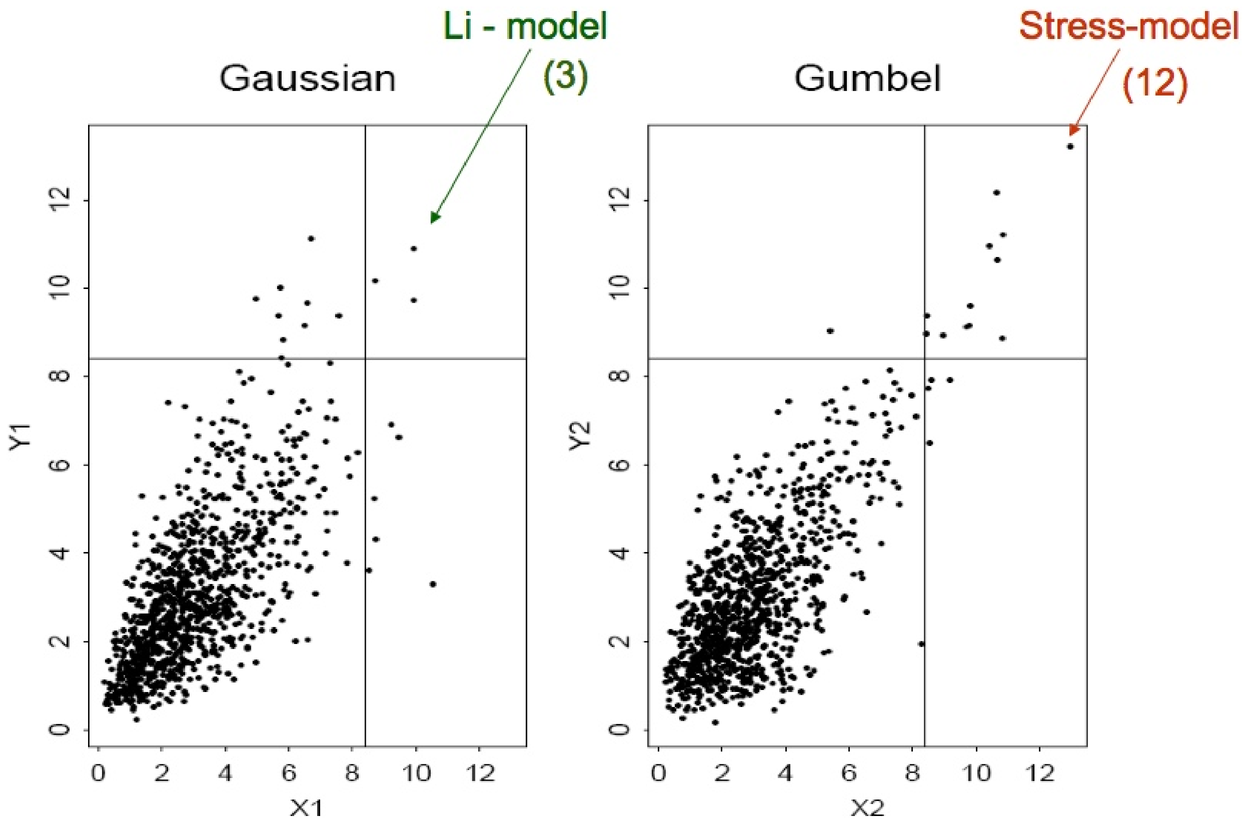
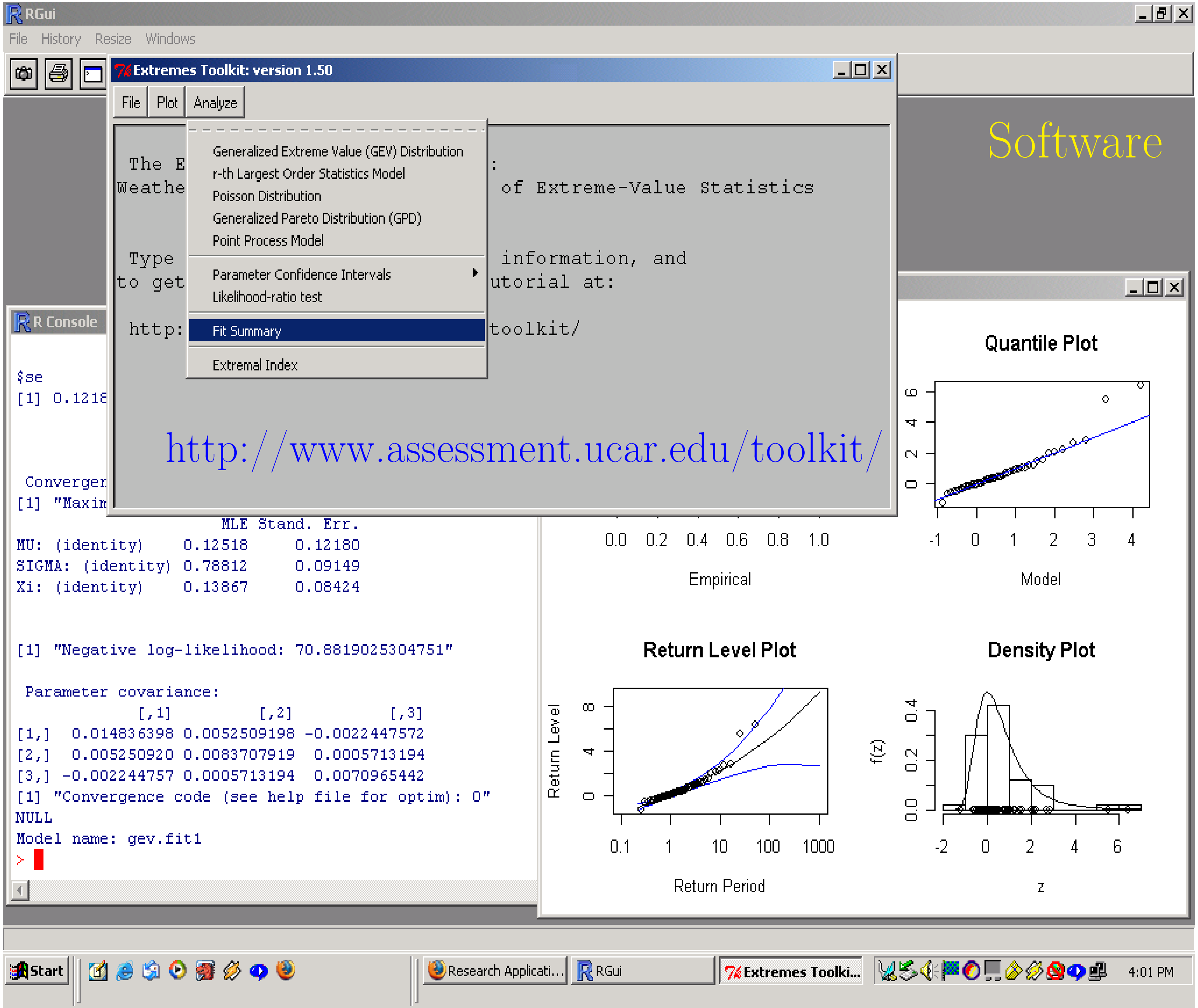


FIGURE 1. 1000 random variates from two distributions with identical Gamma(3,1) marginal distributions and identical correlation $\rho = 0.7$, but *different* dependence structures.

Software

http://www.assessment.ucar.edu/toolkit/



Thanks! Questions?