"Statistics for Near Independence in Multivariate Extremes"

A.W. Ledford and J.A. Tawn

Discussion by Dan Cooley
NCAR Extremes Reading Group
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Forward

These two Ledford and Tawn articles are a precursor to the Heffernan and Tawn paper we looked at several weeks ago. The first(?) alternative to traditional multivariate extreme value theory.

- Review of classical multivariate extreme value theory
- Ledford and Tawn article 1
- A few notes on article 2

Results from Univariate EVT

- $M_t = \max_{i=1,\ldots,t} X_i$.
- A dist. G is *max stable* if $\mathbb{P}\left(\frac{M_t - b_t}{a_t} \leq x\right) = \mathbb{P}(X \leq x) \Leftrightarrow G^t(a_t x + b_t) = G(x)$.
- $F^t(a_t x + b_t) \to G(x), \Rightarrow G$ is max stable
- All univariate max stable distributions are GEV.

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Max Stable Example: Unit Fréchet

$$\mathbb{P}\{X_i \le x\} = F(x) = exp\{-x^{-1}\}$$

Normalizing constants: $a_t = t$, $b_t = 0$

$$\mathbb{P}\{M_t/t \le x\} = F^t(tx) = \left[exp\left\{-\frac{1}{tx}\right\}\right]^t = exp\left\{-\frac{t}{tx}\right\} = exp\{-x^{-1}\}$$

Regular Variation

"Regularly varying functions are those functions which behave asymptotically like power functions".

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{-\alpha}$$

Heavy-tailed random variables have regularly varying tails.
Example: Unit Fréchet

$$\begin{aligned} \lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty} \frac{1-exp\{-(tx)^{-1}\}}{1-exp\{-(t)^{-1}\}} \\ &= \lim_{t\to\infty} \frac{(tx)^{-1}+O(t^{-2})}{t^{-1}+O(t^{-2})} \\ &= \frac{1}{x} \quad \text{(e.g. } 1-F(x) \in RV_1\text{)} \end{aligned}$$

If $U \in RV_\alpha$ then $U(x) = \mathcal{L}(x)x^{-\alpha}$.

Multivariate EVT:

$\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ iid F, $\boldsymbol{X}_i = [X_{i,1}, \ldots, X_{i,d}]^T$

- $\boldsymbol{M}_t = [\max_{i=1,\ldots,t} X_{i,1}, \ldots, \max_{i=1,\ldots,t} X_{i,d}]$.
- A dist. G is max stable if $\mathbb{P}\left(\frac{\boldsymbol{M}_t - \boldsymbol{b}_t}{\boldsymbol{a}_t} \leq \boldsymbol{x}\right) = \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{x}) \Leftrightarrow G^t(\boldsymbol{a}_t\boldsymbol{x} + \boldsymbol{b}_t) = G(\boldsymbol{x})$.
- $F^t(\boldsymbol{a}_t\boldsymbol{x} + \boldsymbol{b}_t) \to G(\boldsymbol{x}) \Rightarrow G$ is max stable [deHaan/Resnick, 1977].
- Q: Can the multivariate EVD be characterized?
  A: Yes, but not easily.

Multivariate EVD's: Characterization 1

WLOG, assume G is a d-dimensional EVD with unit Fréchet margins [Resnick, 1987]. Let $E = [0, \infty]^d \setminus \{\mathbf{0}\}$. Then:

$$G(\boldsymbol{x}) = \exp\left(-\left[\int_0^1 \max\left(\frac{w_1}{x_1}, \ldots, \frac{w_d}{x_d}\right) dH(w)\right]\right),$$

where

$$\sum_{j=1}^{d} w_j = 1,$$

and H is a finite measure on $B = \{\boldsymbol{x} \in E : ||\boldsymbol{x}|| = 1\}$ such that

$$\int_0^1 w dH(w) = 1.$$

Multivariate EVD's: Characterization 2

Again, assume G is a d-dimensional EVD with unit Fréchet margins. Let $E = [0, \infty]^d \setminus \{\mathbf{0}\}$. Then, there exists a non-homogeneous Poisson process on E with intensity measure

$$\mu \left\{ \boldsymbol{x} \in E : ||\boldsymbol{x}|| > r, \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in A \right\} = \frac{H(A)}{r},$$

such that

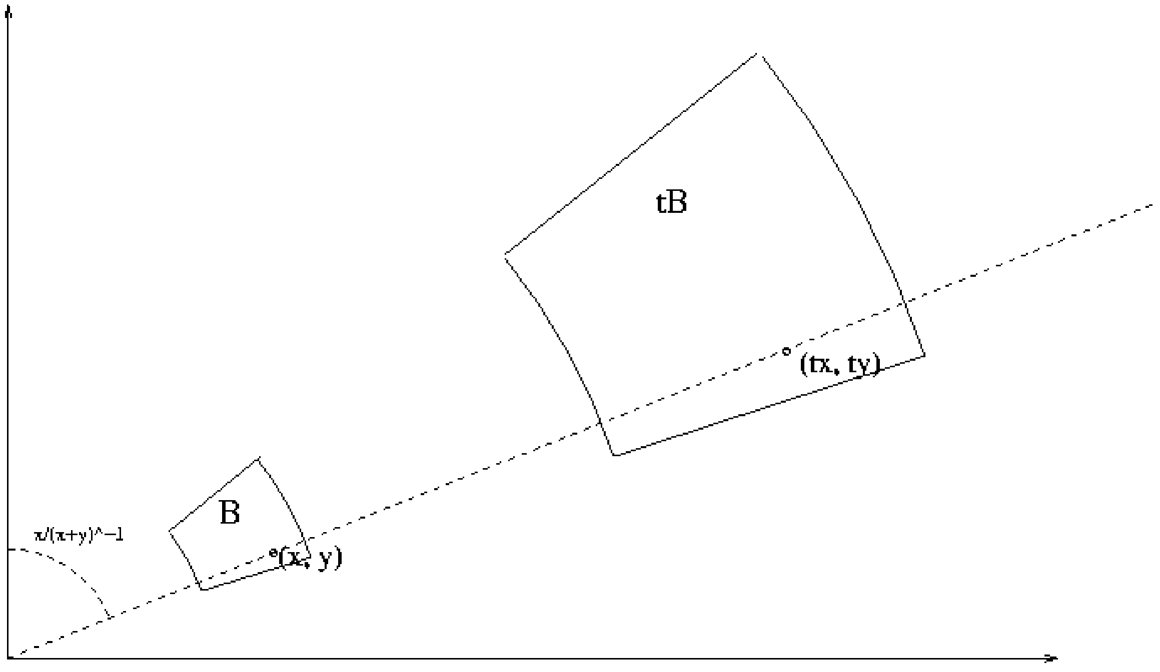
$$G(\boldsymbol{x}) = \exp\left(-\mu\{(\mathbf{0}, \boldsymbol{x}]^c\}\right).$$

Idea: The value of $G(\boldsymbol{x})$ depends only on the "radius" ($||\boldsymbol{x}||$) and the "angular" ($\frac{\boldsymbol{x}}{||\boldsymbol{x}||}$).

Where does characterization 2 come from?

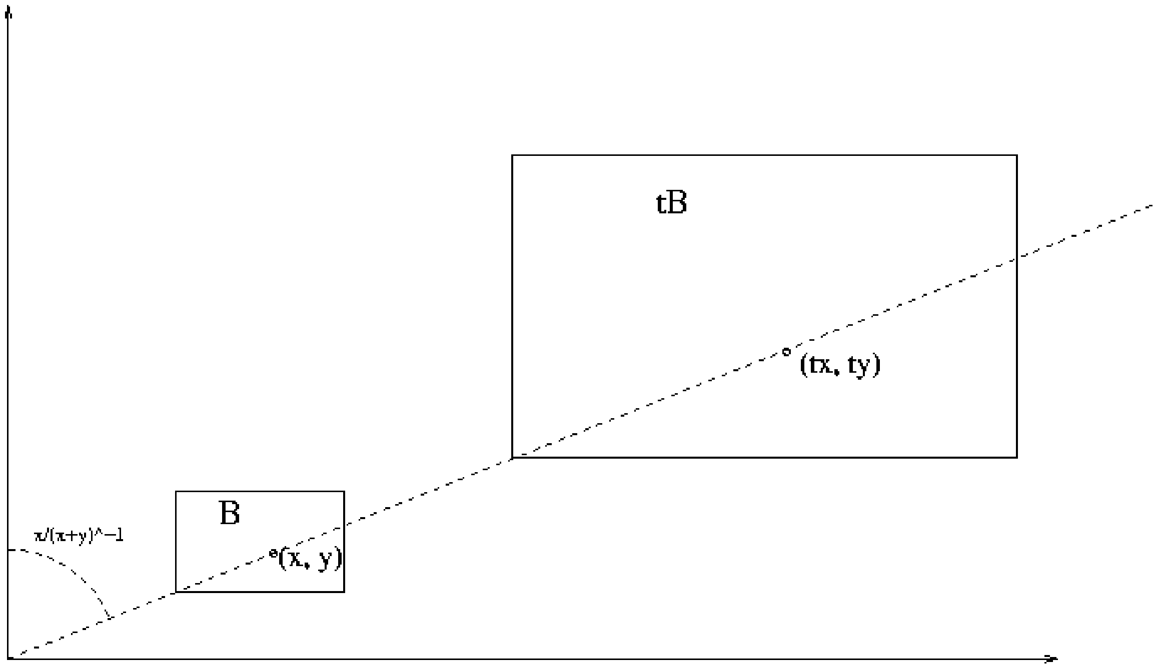
$$\mu\left\{\boldsymbol{x} \in E : ||\boldsymbol{x}|| > r, \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in A\right\} = \frac{H(A)}{r},$$

Given a set B such that $B = \left\{\boldsymbol{x} : \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in A, ||\boldsymbol{x}|| \in (a, b)\right\}$, measure is such that $\mu\{B\} = t\mu\{tB\}$.



Where does characterization 2 come from?

Measure theory: Works for all "shapes" B.



With such a measure, we get max stability:

$$G^t(t\boldsymbol{x}) = [\exp(-\mu\{(\mathbf{0}, t\boldsymbol{x}]^c\})]^t = \exp(-t\mu\{(\mathbf{0}, \boldsymbol{t}x]^c\}) = G(x)$$

Multivariate regular variation

...♪ Kinda goes something like this ♪...

$$\mu_x\{tA\} = \frac{\mathbb{P}(x^{-1}\boldsymbol{X} \in tA)}{\mathbb{P}(||\boldsymbol{X}|| > tx)} \frac{\mathbb{P}(||\boldsymbol{X}|| > tx)}{\mathbb{P}(||\boldsymbol{X}|| > x)} \to \mu(A)t^{-\alpha}$$

- Still has angular idea
- Fréchet case: $\alpha = 1$

Asymptotic Independence

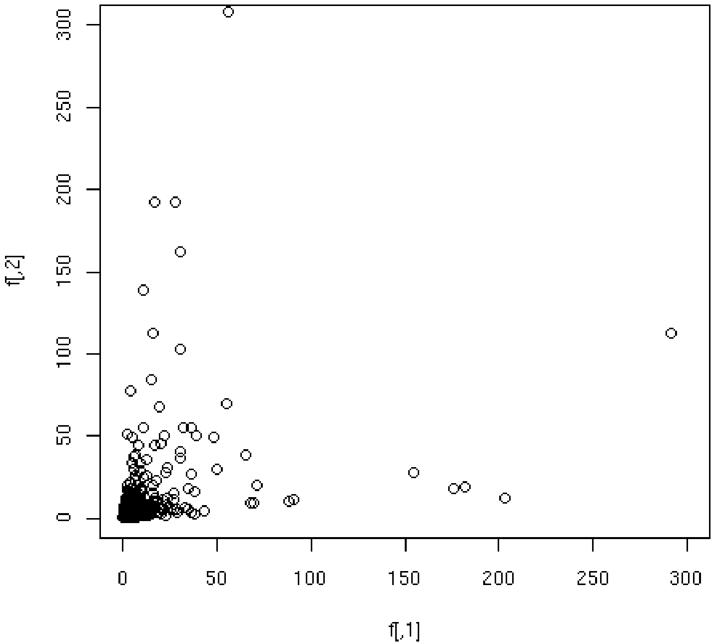
Let (X_1, X_2) be a bivariate random variable with common marginals. Then X_1 and X_2 are *asymptotically independent* if

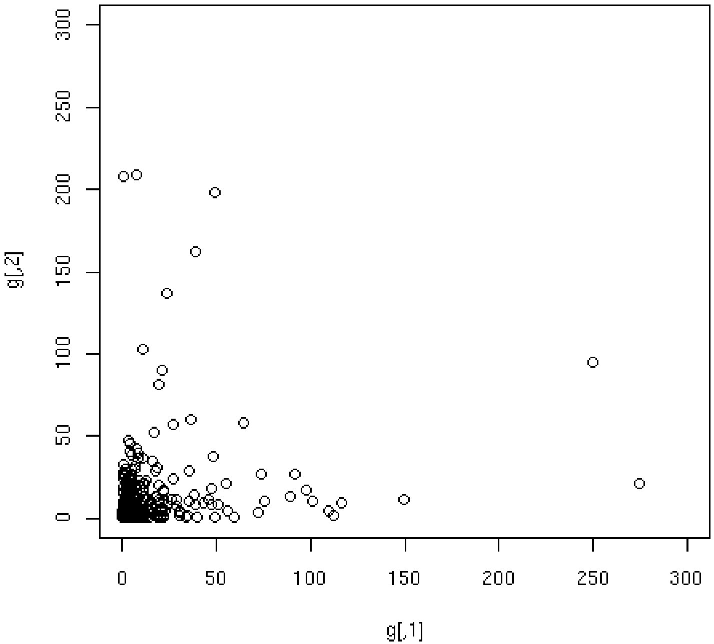
$$\lim_{u \to \infty} \mathbb{P}\{X_1 > u | X_2 > u\} = 0.$$

Example: (X_1, X_2) bivariate normal with $\rho < 1$

Q: How best to handle the middle ground between asymptotically dependent and complete independence?

Example





L&T: 1. Introduction

- Univariate review of threshold exceedance models (GPD and PP).
- Multivariate threshold models:
  - point process model of de Haan (1985)
  - model of Smith, Tawn, and Coles (1997)
- Asymptotic independence
- Problems with above approaches when components are (asymptotically) independent

L&T: 2. Multivariate Threshold Model

- Equations 2.2 and 2.3 ??
- Equations 2.5 and 2.8
- Asymptotic independence case

L&T: 3. Inference for the Model

- Develop censored likelihood (3.2)
- Assume a logistic dependence structure
- Develop score statistic under assumption of independence (3.4)
- Proposition 1

L&T: 4. Variable Thresholds

Similar result holds

L&T: 5. Inference for Asymptotic Independence

- Proposition 2
- η introduced
- Examples
- Estimation for η

L&T: 6. Applications

- Simulations
- Two Examples (wave/surge and wind/rain)

Dan's simulated data

100 points of largest radius used, $X_{min} = min(X_1, X_2)$, GEV fit to X_{min}, $\widehat{\eta} = \widehat{\xi}$

	Normal ($\rho = .8$, $Fr(1)$ Marg.)	Logistic ($\alpha = .7$)
η	.9	1
$\widehat{\eta}$	0.684	0.889
SE	0.106	0.148

L&T II

- From: $\mathbb{P}(Z_1 > r, Z_2 > r) \sim \mathcal{L}(r) r^{-1/\eta}$
  To: $\mathbb{P}(Z_1 > z_1, Z_2 > z_2) \sim \mathcal{L}_1(z_1, z_2) z_1^{-c_1} z_2^{-c_2} + \ldots$
  where $c_1 + c_2 = 1/\eta$
- Make some assumptions, develop a likelihood
- Develop techniques for fitting a model (pretty cool results)
- Point process representation for asymptotic independence

References

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